Second-Order Cone Program (SOCP) Detection and Transformation Algorithms for Optimization Software

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Second-Order Cone Programs (SOCPs)

- Can be written as a quadratic program
- Not positive semi-definite
- Convex
- Efficiently solvable with interior-point methods

SOCP general form:

$$\text{minimize } f^T x$$

$$\text{subject to } \|A_i x + b_i\|^2 \leq (c_i^T x + d_i)^2 \ \forall \ i$$

$$c_i^T x + d_i \geq 0 \ \forall \ i$$

Introduction

Previous situation:

- SOCPs can be written in numerous equivalent forms
- The form a modeler wants to use may not be the form a solver accepts
- Converting the problem for a particular interior-point solver is tedious and error-prone

Ideal situation:

- Write in modeler's form in a general modeling language
- Automatically transform to a standard quadratic formulation
- Transform as necessary for each SOCP solver

Motivating Example

$$\text{minimize } \sqrt{(x+2)^2 + (y+1)^2} + \sqrt{(x+y)^2}$$

Ampl model:

```
var x;
var y;
minimize objective:
   sqrt((x+2)^2+(y+1)^2)+sqrt((x+y)^2);
```

CPLEX 12.2.0.0: at2372.nl contains a nonlinear objective.
KNITRO 6.0.0: Current feasible solution estimate cannot be improved.
objective 2.12251253;
30 iterations; 209 function evaluations

Motivating Example

Original:

$$\text{minimize } \sqrt{(x+2)^2 + (y+1)^2} + \sqrt{(x+y)^2}$$

Transformed:

$$\text{minimize } u + v$$

$$(x+2)^2 + (y+1)^2 \leq u^2$$

$$(x+y)^2 \leq v^2$$

$$u, v \geq 0$$

Motivating Example

Ampl model:

```
var x; var y;
var u >= 0; var v >= 0;
minimize obj:  u+v;
s.t.  C1:  (x+2)^2+(y+1)^2 <= u^2;
s.t.  C2:  (x+y)^2 <= v^2;
```

CPLEX 12.2.0.0: QP Hessian is not positive semi-definite.
KNITRO 6.0.0: Locally optimal solution.
objective 2.122027399;
3161 iterations; 3276 function evaluations

Motivating Example

Original:

$$\text{minimize } \sqrt{(x+2)^2 + (y+1)^2} + \sqrt{(x+y)^2}$$

Transformed:

$$\text{minimize } u + v$$

$$r^2 + s^2 \leq u^2$$

$$t^2 \leq v^2$$

$$x + 2 = r$$

$$y + 1 = s$$

$$x + y = t$$

$$u, v \geq 0$$

Motivating Example

Ampl model:

```
var x; var y;
var u >= 0; var v >= 0;
var r; var s; var t;
minimize obj:  u+v;
s.t.  C1:  r^2+s^2 <= u^2;
s.t.  C2:  t^2 <= v^2;
s.t.  C3:  x+2 = r;
s.t.  C4:  y+1 = s;
s.t.  C5:  x+y = t;
```

CPLEX 12.2.0.0: primal optimal; objective 2.121320344
5 barrier iterations
KNITRO 6.0.0: Locally optimal solution.
objective 2.122027305;
3087 iterations; 3088 function evaluations

Generally Accepted SOCP Form

SOCP general form:

$$\text{minimize } f^T x$$

$$\text{subject to } \|A_i x + b_i\| \leq c_i^T x + d_i \ \forall i$$

where

- $x \in \mathbb{R}^n$ is the variable
- $f \in \mathbb{R}^n$
- $A_i \in \mathbb{R}^{m_i,n}$
- $b_i \in \mathbb{R}^{m_i}$
- $c_i \in \mathbb{R}^n$
- $d_i \in \mathbb{R}$

Standard Quadratic Form

Objective: $a_1x_1 + \cdots + a_nx_n$

Constraints:

Quadratic Cone: $b_1x_1^2 + \cdots + b_nx_n^2 - b_0x_0^2 \leq 0$

where $b_i \geq 0 \; \forall \; i$, $x_0 \geq 0$

Rotated Quadratic Cone: $c_2x_2^2 + \cdots + c_nx_n^2 - c_1x_0x_1 \leq 0$

where $c_i \geq 0 \; \forall \; i$, $x_0 \geq 0$, $x_1 \geq 0$

Linear Inequality: $d_0 + d_1x_1 + \cdots + d_nx_n \leq 0$

Linear Equality: $e_0 + e_1x_1 + \cdots + e_nx_n = 0$

Variable: $k_L \leq x$

Detection Example

$$\text{minimize } \sqrt{(x+2)^2+(y+1)^2}+\sqrt{(x+y)^2}$$

Ampl model:

```
var x;
var y;
minimize objective:
   sqrt((x+2)^2+(y+1)^2)+sqrt((x+y)^2);
```

First Case: Sum and Max of Norms

Any combination of

- sum,
- max, and
- constant multiple

of norms can be represented as a SOCP.

Sum of Norms

$$\text{minimize} \ \sum_{i=1}^{p} \|F_i x + g_i\|$$

$$\Longrightarrow$$

$$\begin{aligned}
\text{minimize} \ & \sum_{i=1}^{p} y_i \\
\text{subject to} \ & \sum_{j=1}^{q_i} u_{ij}^2 - y_i^2 \leq 0, \ \ i = 1..p \\
& (F_i x + g_i)_j - u_{ij} = 0, \ \ i = 1..p, \ \ j = 1..q_i \\
& y_i \geq 0, \ \ i = 1..p
\end{aligned}$$

Max of Norms

$$\text{minimize} \max_{i=1..p} \|F_i x + g_i\|$$

$$\Longrightarrow$$

$$\begin{aligned}
&\text{minimize } y \\
&\text{subject to } \sum_{j=1}^{q_i} u_{ij}^2 - y^2 \le 0, \; i = 1..p \\
&\qquad (F_i x + g_i)_j - u_{ij} = 0, \; i = 1..p, \; j = 1..q_i \\
&\qquad y_i \ge 0, \; i = 1..p
\end{aligned}$$

Combination

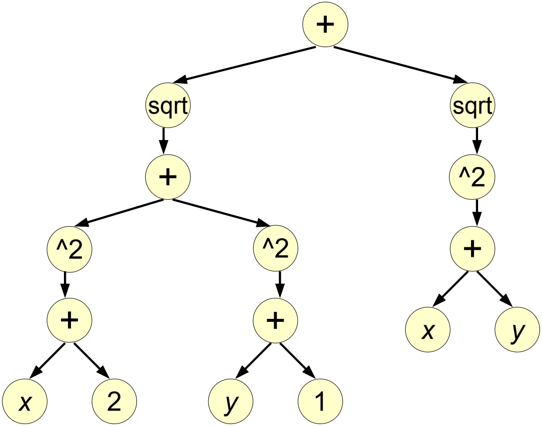
$$\text{minimize } 4\max\{3\|F_1x+g_1\|+2\|F_2x+g_2\|, 7\|F_3x+g_3\|\}$$

$\Longrightarrow$

$$\begin{aligned}
\text{minimize } & 4y \\
\text{subject to } & 3u_1+2u_2-y \le 0 \\
& 7u_3-y \le 0 \\
& \sum_{j=1}^{q_i} v_{ij}^2-u_i^2 \le 0,\ \ i=1,2,3 \\
& (F_ix+g_i)_j-v_{ij}=0,\ \ i=1,2,3,\ \ j=1..q_i \\
& u_i \ge 0,\ \ i=1,2,3
\end{aligned}$$

Expression Tree Example

$$\sqrt{(x+2)^2+(y+1)^2}+\sqrt{(x+y)^2}$$



Sum and Max of Norms (SMN) Detection Function

Detection Rules for SMN:

Constant: $f(x) = c$ is SMN.
Variable: $f(x) = x_i$ is SMN.
Sum: $f(x) = \sum_{i=1}^{n} f_i(x)$ is SMN if all the children f_i are SMN.
Product: $f(x) = cg(x)$ is SMN if c is a positive constant and g is SMN.
Maximum: $f(x) = \max\{f_1(x), \ldots, f_n(x)\}$ is SMN if all the children f_i are SMN.
Square Root: $f(x) = \sqrt{g(x)}$ is SMN if g is NS.

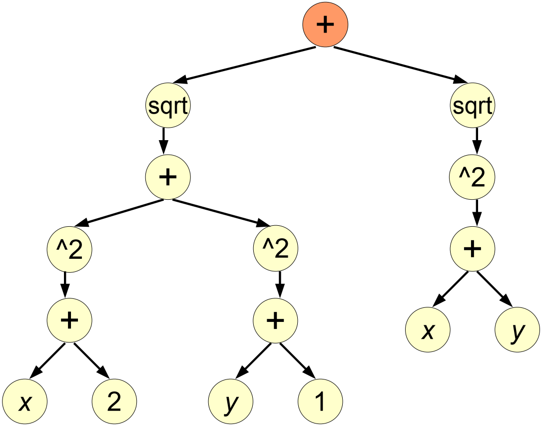
Norm Squared (NS) Detection Function

Detection Rules for NS:

Constant: $f(x) = c$ is NS if $c \geq 0$.
Sum: $f(x) = \sum_{i=1}^{n} f_i(x)$ is NS if all the children f_i are NS.
Product: $f(x) = cg(x)$ is NS if c is a positive constant and g is NS.
Squared: $f(x) = g(x)^2$ is NS if g is linear.
Maximum: $f(x) = \max\{f_1(x), \ldots, f_n(x)\}$ is NS if all the children f_i are NS.

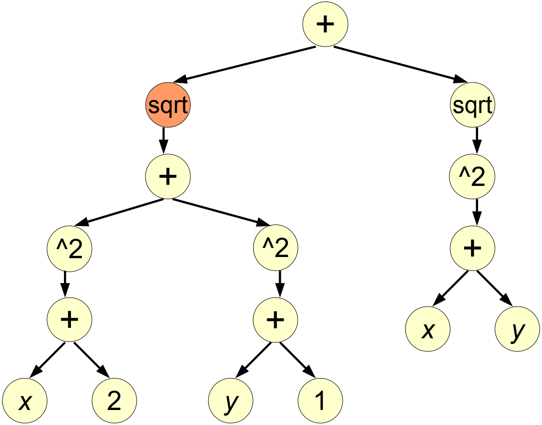
Detection Example

Sum: $f(x) = \sum_{i=1}^{n} f_i(x)$ is SMN if all the children f_i are SMN.



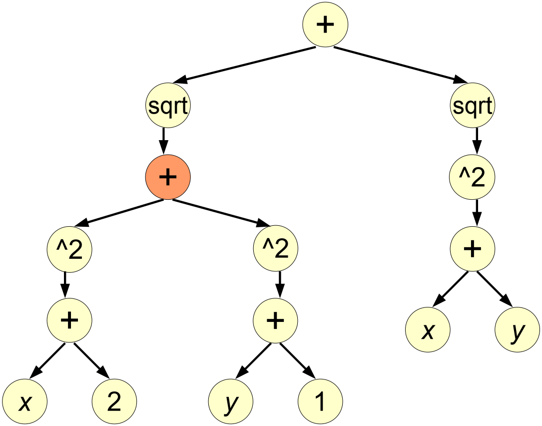
Detection Example

Square Root: $f(x) = \sqrt{g(x)}$ is SMN if $g(x)$ is NS.



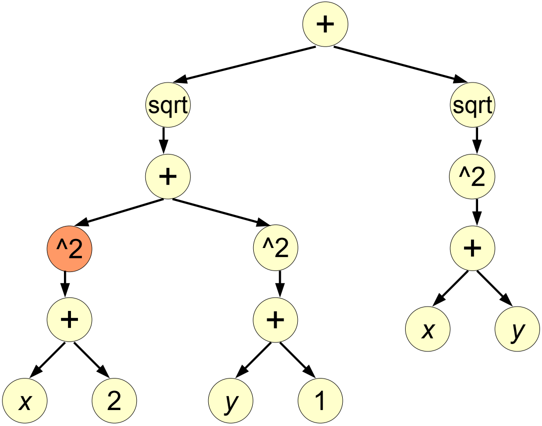
Detection Example

Sum: $f(x) = \sum_{i=1}^{n} f_i(x)$ is NS if all the children f_i are NS.



Detection Example

Squared: $f(x) = g(x)^2$ is NS if g is linear.



Transformation Process

1) Determine objective or constraint type with detection rules

2) Apply corresponding transformation algorithm

- Separate algorithm, starts at root
- Uses no information from detection
- Creates new variables and constraints
- New constraints are formed by adding terms to functions

Transformation Conventions

x: vector of variables in the original formulation
v: vector of variables in the original formulation and variables created during transformation
$f(x), g(x), h(x)$: functions from the original formulation

Functions created during transformation:
$o(v)$: objectives (linear)
$\ell(v)$: linear inequalities
$e(v)$: linear equalities
$q(v)$: quadratic cones
$r(v)$: rotated quadratic cones
$c(v)$: expressions that could fit in multiple categories

Constraint Building Example

Step 1: $\ell_1(v) := 3x_1 + 2$

Step 2: $\ell_1(v) := \ell_1(v) + v_3$

Step 3: $\ell_1(v) \leq 0$

Result: $3x_1 + 2 + v_3 \leq 0$

Transformation Functions

newvar(b)
  $n++$
  Introduce new variable v_n to variable vector v
  if b is specified
    Set lower bound of v_n to b
  else
    Set lower bound of v_n to $-\infty$

newfunc(c)
  m_c++
  Introduce new objective or constraint function of type c
  $c_{m_c}(v) := 0$

transformSMN

transformSMN($f(x), c(v), k$)
switch
case $f(x) = g(x) + h(x)$
transformSMN($g(x), c(v), k$)
transformSMN($h(x), c(v), k$)
case $f(x) = \sum_i f_i(x)$
transformSMN($f_i(x), c(v), k$) $\forall\ i$
case $f(x) = \alpha g(x)$
transformSMN($g(x), c(v), k\alpha$)

transformSMN

case $f(x) = \max_i f_i(x)$

  newvar()

  $c(v) := c(v) + kv_n$

  newfunc(ℓ): $\ell_{m_\ell}(v) := -v_n$

  for $i \in I$

    transformSMN($f_i(x), \ell_{m_\ell}(v), 1$)

case $f(x) = \sqrt{g(x)}$

  newvar(0)

  $c(v) := c(v) + kv_n$

  newfunc(q): $q_{m_q}(v) := -v_n^2$

  transformNS($g(x), q_{m_q}(v), 1$)

transformNS

transformNS($f(x), q(v), k$)
  switch
    case $f(x) = g(x) + h(x)$
      transformNS($g(x), c(v), k$)
      transformNS($h(x), c(v), k$)
    case $f(x) = \sum_i f_i(x)$
      transformNS($f_i(x), c(v), k$) $\forall\ i$
    case $f(x) = \alpha g(x)$
      transformNS($g(x), c(v), k\alpha$)

transformNS

case $f(x) = g(x)^2$
  newvar()
  $q(v) := q(v) + kv_n^2$
  newfunc(e): $e_{m_e}(v) := g(x) - v_n$
case $f(x) = \max_i f_i(x)$
  newvar()
  $q(v) := q(v) + kv_n^2$
  for $i \in I$
    newfunc(q): $q_{m_q}(v) := -v_n^2$
    transformNS$(f_i(x), q_{m_q}(v), 1)$

Transformation Example

$$f(x) = \sqrt{(x+2)^2 + (y+1)^2} + \sqrt{(x+y)^2}$$

$f(x)$ is SMN $\Rightarrow$ apply corresponding transformation algorithm

Set all index variables to 0
$o(v) := 0$
transformSMN$(f(x), o(v), 1)$

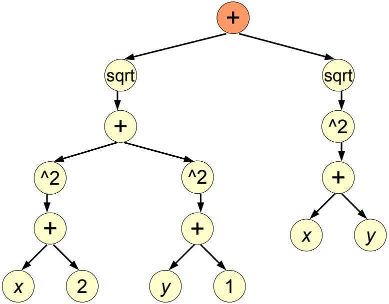
Transformation Example

$$f(x) = \sqrt{(x+2)^2 + (y+1)^2} + \sqrt{(x+y)^2}$$

case $f(x) = g(x) + h(x)$

transformSMN$(g(x), o(v), 1)$

transformSMN$(h(x), o(v), 1)$



Transformation Example

$$f(x) = \sqrt{(x+2)^2 + (y+1)^2}$$

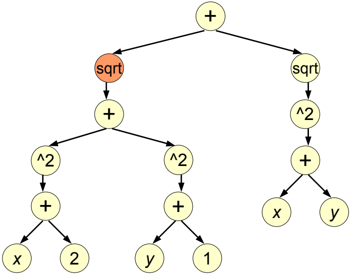
case $f(x) = \sqrt{g(x)}$

newvar(0)

$o(v) := o(v) + v_1$

newfunc(q): $q_1(v) := -v_1^2$

transformNS($g(x), q_1(v), 1$)



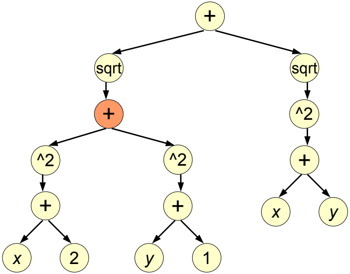
Transformation Example

$$f(x) = (x+2)^2 + (y+1)^2$$

case $f(x) = g(x) + h(x)$

transformNS$(g(x), q_1(v), k)$

transformNS$(h(x), q_1(v), k)$



Transformation Example

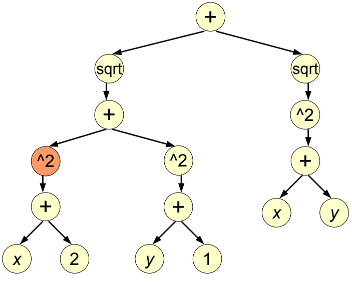
$$f(x) = (x+2)^2$$

case $f(x) = g(x)^2$

newvar()

$q_1(v) := q_1(v) + v_2^2$

newfunc(e): $e_1(v) := g(x) - v_2$



Current Functions

$$o(v) = v_1$$
$$q_1(v) = v_2^2 - v_1^2$$
$$e_1(v) = x + 2 - v_2$$
$$v_1 \geq 0$$

Final Functions

$$o(v) = v_1 + v_4$$

$$q_1(v) = v_2^2 + v_3^2 - v_1^2 \leq 0$$

$$e_1(v) = x + 2 - v_2 = 0$$

$$e_2(v) = y + 1 - v_3 = 0$$

$$q_2(v) = v_5^2 - v_4^2 \leq 0$$

$$e_3(v) = x + y - v_5 = 0$$

$$v_1 \geq 0$$

$$v_4 \geq 0$$

Motivating Example

Original:

$$\text{minimize } \sqrt{(x+2)^2+(y+1)^2}+\sqrt{(x+y)^2}$$

Transformed:

$$\text{minimize } u+v$$

$$r^2+s^2 \leq u^2$$

$$t^2 \leq v^2$$

$$x+2=r$$

$$y+1=s$$

$$x+y=t$$

$$u,v \geq 0$$

Other Objective Forms

- Norm Squared:

$$\text{minimize } \sum_{i=1}^{p} c_i(a_i x + b_i)^2$$

- Fractional:

$$\text{minimize } \sum_{i=1}^{p} \frac{c_i\|F_i x + g_i\|^2}{a_i x + b_i}$$

  where $a_i x + b_i > 0 \ \forall \ i$

- Logarithmic Chebyshev:

$$\text{minimize } \max_{i=1..p} |\log(a_i x) - \log(b_i)|$$

  where $a_i x > 0$

Other Objective Forms

- Product of Positive Powers:

$$\text{maximize} \ \prod_{i=1}^{p}(a_i x + b_i)^{\alpha_i}$$

where $\alpha_i > 0$, $\alpha_i \in \mathbb{Q}$, $a_i x + b_i \geq 0$

- Product of Negative Powers:

$$\text{minimize} \ \prod_{i=1}^{p}(a_i x + b_i)^{-\pi_i}$$

where $\pi_i > 0$, $\pi_i \in \mathbb{Q}$, $a_i x + b_i \geq 0$

- Combinations of these forms made by sum, max, and positive constant multiple, except Log Chebyshev and some cases of Product of Positive Powers. Example:

$$\text{minimize} \ \max\{\sum_{i=1}^{p}(a_i x + b_i)^2, \sum_{j=1}^{q}\frac{\|F_j x + g_j\|^2}{y_j}\} + \prod_{k=1}^{r}(c_k x)^{-\pi_k}$$

Constraint Forms

- Sum and Max of Norms:

$$\sum_{i=1}^{p} c_i \|F_i x + g_i\| \leq ax + b$$

- Norm Squared:

$$\sum_{i=1}^{p} c_i (a_i x + b_i)^2 \leq c_0 (a_0 x + b_0)^2$$

where $c_0 \geq 0,\ a_0 x + b_0 \geq 0$

- Fractional:

$$\sum_{i=1}^{p} \frac{k_i \|F_i x + g_i\|^2}{a_i x + b_i} \leq cx + d$$

where $a_i x + b_i > 0\ \forall\ i$

Constraint Forms

- Product of Positive Powers:

$$\sum_j - \prod_i (a_{ji}x + b_{ji})^{\pi_{ji}} \leq cx + d$$

where $a_{ji}x + b_{ji} \geq 0$, $\pi_{ji} > 0$, $\sum_i \pi_{ji} \leq 1 \forall j$

- Product of Negative Powers:

$$\sum_j \prod_i (a_{ji}x + b_{ji})^{-\pi_{ji}} \leq cx + d$$

where $a_{ji}x + b_{ji} \geq 0$, $\pi_{ji} > 0$

- Combinations of these forms made by sum, max, and positive constant multiple

Conclusion

- Implementation for AMPL and several solvers
- Paper documenting algorithms
- Extend to functions not included in AMPL

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Thank You

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