Alternatives for Scripting in Conjunction with an Algebraic Modeling Language for Optimization

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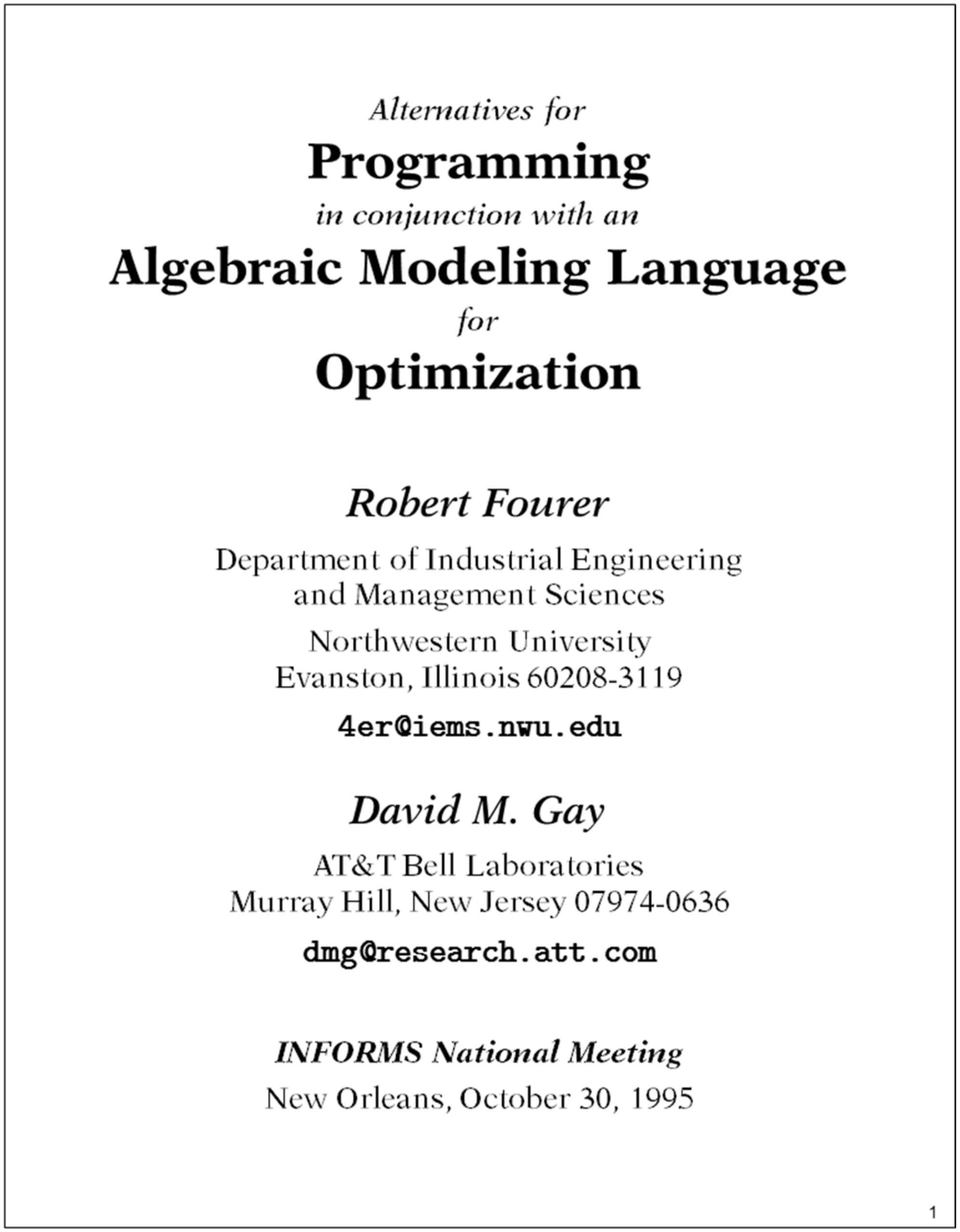
Alternatives for Scripting in Conjunction with an Algebraic Modeling Language for Optimization

Optimization modeling languages are fundamentally declarative, yet successful languages also offer ways to write scripts or programs. What can scripting in a modeling language offer in comparison to modeling in a general-purpose scripting language? Some answers will be suggested through diverse examples in which the AMPL modeling language is applied to parametric analysis, solution generation, heuristic optimization, pattern enumeration, and decomposition. Concluding comments will touch on the complexity of scripts seen in practical applications, and on prospects for further improvements.



Alternatives for

Programming

in conjunction with an

Algebraic Modeling Language

for

Optimization

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Topics: Introduction to AMPL

The optimization modeling cycle

Optimization modeling languages

Example: multicommodity transportation

- Mathematical formulation
- AMPL formulation
- AMPL solution

Topics: Scripting in AMPL

1: *Parametric analysis*

2: *Solution generation*

- **a:** *via cuts*
- **b:** *via solver*

3: *Heuristic optimization*

4: *Pattern generation*

5: *Decomposition*

Scripts in practice . . .

Prospective improvements . . .

The Optimization Modeling Cycle

Steps

- Communicate with problem owner
- Build model
- Prepare data
- Generate optimization problem
- Submit problem to solver
  - CPLEX, Gurobi, KNITRO, CONOPT, MINOS, . . .
- Report & analyze results
- ***Repeat!***

Goals

- Do this quickly and reliably
- Get results before client loses interest
- Deploy for application

What Makes This Hard?

“We do not feel that the linear programming user’s most pressing need over the next few years is for a new optimizer that runs twice as fast on a machine that costs half as much (although this will probably happen). Cost of optimization is just not the dominant barrier to LP model implementation.

“The process required to manage the data, formulate and build the model, report on and analyze the results costs far more, and is much more of a barrier to effective use of LP, than the cost/performance of the optimizer.”

Krabek, Sjoquist, Sommer,
“The APEX Systems: Past and Future.”
***SIGMAP Bulletin* 29 (April 1980) 3-23.**

Optimization Modeling Languages

Two forms of an optimization problem

- Modeler's form
  - Mathematical description, easy for people to work with
- Algorithm's form
  - Explicit data structure, easy for solvers to compute with

Idea of a modeling language

- A computer-readable modeler's form
  - You write optimization problems in a modeling language
  - Computers translate to algorithm's form for solution

Advantages of a modeling language

- Faster modeling cycles
- More reliable modeling and maintenance

Algebraic Modeling Languages

Formulation concept

- Define data in terms of sets & parameters
  - Analogous to database keys & records
- Define decision variables
- Minimize or maximize a function of decision variables
- Subject to equations or inequalities that constrain the values of the variables

Advantages

- Familiar
- Powerful
- Implemented

The AMPL Modeling Language

Features

- Algebraic modeling language
- Variety of data sources
- Connections to all solver features
- Interactive and scripted control

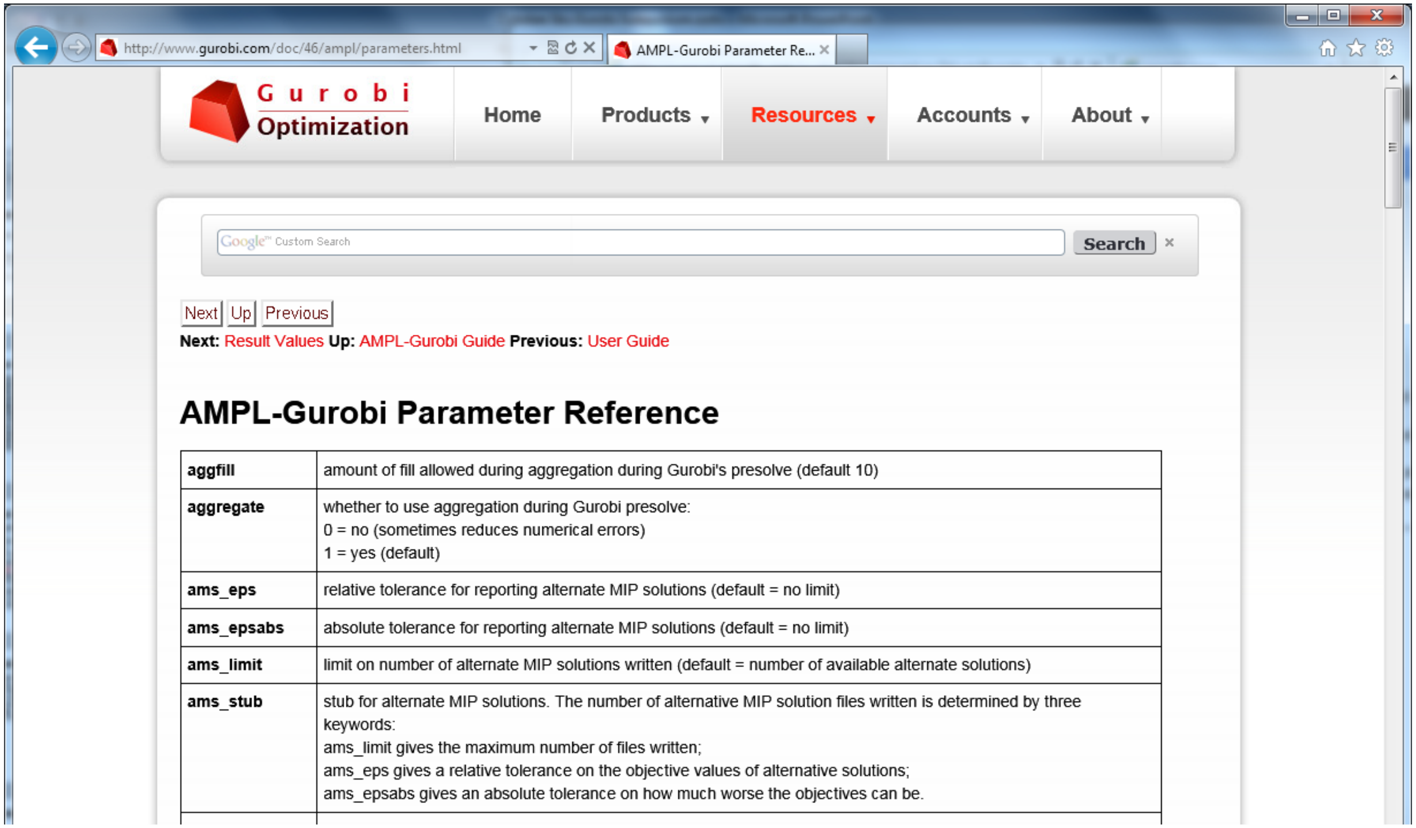
Advantages

- Powerful, general expressions
- Natural, easy-to-learn design
- Efficient processing scales well with problem size

AMPL with Gurobi

Features

- Detection of all supported problem types
- Access to all algorithm & display options



Next: Result Values **Up:** AMPL-Gurobi Guide **Previous:** User Guide

AMPL-Gurobi Parameter Reference

aggfill	amount of fill allowed during aggregation during Gurobi's presolve (default 10)
aggregate	whether to use aggregation during Gurobi presolve: 0 = no (sometimes reduces numerical errors) 1 = yes (default)
ams_eps	relative tolerance for reporting alternate MIP solutions (default = no limit)
ams_epsabs	absolute tolerance for reporting alternate MIP solutions (default = no limit)
ams_limit	limit on number of alternate MIP solutions written (default = number of available alternate solutions)
ams_stub	stub for alternate MIP solutions. The number of alternative MIP solution files written is determined by three keywords: ams_limit gives the maximum number of files written; ams_eps gives a relative tolerance on the objective values of alternative solutions; ams_epsabs gives an absolute tolerance on how much worse the objectives can be.

Introductory Example

Multicommodity transportation . . .

- Products available at factories
- Products needed at stores
- Plan shipments at lowest cost

. . . with practical restrictions

- Cost has fixed and variable parts
- Shipments cannot be too small
- Factories cannot serve too many stores

Multicommodity Transportation

Given

O	Set of origins (factories)
D	Set of destinations (stores)
P	Set of products

and

a_{ip}	Amount available, for each $i \in O$ and $p \in P$
b_{jp}	Amount required, for each $j \in D$ and $p \in P$
l_{ij}	Limit on total shipments, for each $i \in O$ and $j \in D$
c_{ijp}	Shipping cost per unit, for each $i \in O$, $j \in D$, $p \in P$
d_{ij}	Fixed cost for shipping any amount from $i \in O$ to $j \in D$
s	Minimum total size of any shipment
n	Maximum number of destinations served by any origin

Multicommodity Transportation

Mathematical Formulation

Determine

X_{ijp} Amount of each $p \in P$ to be shipped from $i \in O$ to $j \in D$

Y_{ij} 1 if any product is shipped from $i \in O$ to $j \in D$
0 otherwise

to minimize

$$\sum_{i \in O} \sum_{j \in D} \sum_{p \in P} c_{ijp} X_{ijp} + \sum_{i \in O} \sum_{j \in D} d_{ij} Y_{ij}$$

Total variable cost plus total fixed cost

Mathematical Formulation

Subject to

$\sum_{j \in D} X_{ijp} \leq a_{ip}$ for all $i \in O$, $p \in P$

Total shipments of product p out of origin i must not exceed availability

$\sum_{i \in O} X_{ijp} = b_{jp}$ for all $j \in D$, $p \in P$

Total shipments of product p into destination j must satisfy requirements

Multicommodity Transportation

Mathematical Formulation

Subject to

$\sum_{p \in P} X_{ijp} \leq l_{ij} Y_{ij}$ for all $i \in O, j \in D$

When there are shipments from origin i to destination j, the total may not exceed the limit, and Y_{ij} must be 1

$\sum_{p \in P} X_{ijp} \geq s Y_{ij}$ for all $i \in O, j \in D$

When there are shipments from origin i to destination j, the total amount of shipments must be at least s

$\sum_{j \in D} Y_{ij} \leq n$ for all $i \in O$

Number of destinations served by origin i
must be as most n

Multicommodity Transportation

AMPL Formulation

Symbolic data

```
set ORIG;   # origins
set DEST;   # destinations
set PROD;   # products

param supply {ORIG,PROD} >= 0;  # availabilities at origins
param demand {DEST,PROD} >= 0;  # requirements at destinations
param limit {ORIG,DEST} >= 0;   # capacities of links

param vcost {ORIG,DEST,PROD} >= 0; # variable shipment cost
param fcost {ORIG,DEST} > 0;       # fixed usage cost

param minload >= 0;             # minimum shipment size
param maxserve integer > 0;     # maximum destinations served
```

Multicommodity Transportation

AMPL Formulation

Symbolic model: variables and objective

```
var Trans {ORIG,DEST,PROD} >= 0;   # actual units to be shipped
var Use {ORIG, DEST} binary;       # 1 if link used, 0 otherwise

minimize Total_Cost:
   sum {i in ORIG, j in DEST, p in PROD} vcost[i,j,p] * Trans[i,j,p]
 + sum {i in ORIG, j in DEST} fcost[i,j] * Use[i,j];
```

$$\sum_{i \in O} \sum_{j \in D} \sum_{p \in P} c_{ijp} X_{ijp} + \sum_{i \in O} \sum_{j \in D} d_{ij} Y_{ij}$$

Multicommodity Transportation

AMPL Formulation

Symbolic model: constraint

```
subject to Supply {i in ORIG, p in PROD}:
   sum {j in DEST} Trans[i,j,p] <= supply[i,p];
```

$$\sum_{j \in D} X_{ijp} \leq a_{ip}, \text{ for all } i \in O, p \in P$$

Multicommodity Transportation

AMPL Formulation

Symbolic model: constraints

```
subject to Supply {i in ORIG, p in PROD}:
   sum {j in DEST} Trans[i,j,p] <= supply[i,p];

subject to Demand {j in DEST, p in PROD}:
   sum {i in ORIG} Trans[i,j,p] = demand[j,p];

subject to Multi {i in ORIG, j in DEST}:
   sum {p in PROD} Trans[i,j,p] <= limit[i,j] * Use[i,j];

subject to Min_Ship {i in ORIG, j in DEST}:
   sum {p in PROD} Trans[i,j,p] >= minload * Use[i,j];

subject to Max_Serve {i in ORIG}:
   sum {j in DEST} Use[i,j] <= maxserve;
```

Multicommodity Transportation

AMPL Formulation

Explicit data independent of symbolic model

```
set ORIG := GARY CLEV PITT ;
set DEST := FRA DET LAN WIN STL FRE LAF ;
set PROD := bands coils plate ;

param supply (tr):  GARY   CLEV   PITT :=
            bands    400    700    800
            coils    800   1600   1800
            plate    200    300    300 ;

param demand (tr):
           FRA   DET   LAN   WIN   STL   FRE   LAF :=
   bands   300   300   100    75   650   225   250
   coils   500   750   400   250   950   850   500
   plate   100   100     0    50   200   100   250 ;

param limit default 625 ;

param minload := 375 ;
param maxserve := 5 ;
```

Multicommodity Transportation

AMPL Formulation

Explicit data (continued)

```
param vcost :=

 [*,*,bands]:  FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY    30   10    8   10   11   71    6
        CLEV    22    7   10    7   21   82   13
        PITT    19   11   12   10   25   83   15

 [*,*,coils]:  FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY    39   14   11   14   16   82    8
        CLEV    27    9   12    9   26   95   17
        PITT    24   14   17   13   28   99   20

 [*,*,plate]:  FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY    41   15   12   16   17   86    8
        CLEV    29    9   13    9   28   99   18
        PITT    26   14   17   13   31  104   20 ;

param fcost:   FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY  3000 1200 1200 1200 2500 3500 2500
        CLEV  2000 1000 1500 1200 2500 3000 2200
        PITT  2000 1200 1500 1500 2500 3500 2200 ;
```

Multicommodity Transportation

AMPL Solution

Model + data = problem instance to be *solved*

```
ampl: model multmipG.mod;
ampl: data multmipG.dat;

ampl: option solver gurobi;

ampl: solve;

Gurobi 5.0.0: optimal solution; objective 235625
394 simplex iterations
46 branch-and-cut nodes

ampl: display Use;

Use [*,*]

:    DET FRA FRE LAF LAN STL WIN  :=
CLEV   1   1   1   0   1   1   0
GARY   0   0   0   1   0   1   1
PITT   1   1   1   1   0   1   0
;
```

Multicommodity Transportation

AMPL Solution

Solver choice independent of model and data

```
ampl: model multmipG.mod;
ampl: data multmipG.dat;

ampl: option solver cplex;

ampl: solve;

CPLEX 12.4.0.0: optimal integer solution; objective 235625
394 MIP simplex iterations
41 branch-and-bound nodes

ampl: display Use;

Use [*,*]

:    DET FRA FRE LAF LAN STL WIN  :=
CLEV   1   1   1   0   1   1   0
GARY   0   0   0   1   0   1   1
PITT   1   1   1   1   0   1   0
;
```

Multicommodity Transportation

AMPL Solution

Examine results

```
ampl: display {i in ORIG, j in DEST}
ampl?    sum {p in PROD} Trans[i,j,p] / limit[i,j];

:       DET     FRA     FRE     LAF     LAN    STL     WIN     :=
CLEV    1       0.6     0.88    0       0.8    0.88    0
GARY    0       0       0       0.64    0      1       0.6
PITT    0.84    0.84    1       0.96    0      1       0
;

ampl: display Max_Serve.body;

CLEV   5
GARY   3
PITT   5
;

ampl: display TotalCost,
ampl?    sum {i in ORIG, j in DEST} fcost[i,j] * Use[i,j];

TotalCost = 235625
sum {i in ORIG, j in DEST} fcost[i,j]*Use[i,j] = 27600
```

Multicommodity Transportation

AMPL “Sparse” Network

Indexed over sets of pairs and triples

```
set ORIG;   # origins
set DEST;   # destinations
set PROD;   # products

set SHIP within {ORIG,DEST,PROD};

            # (i,j,p) in SHIP ==> can ship p from i to j

set LINK = setof {(i,j,p) in SHIP} (i,j);

            # (i,j) in LINK ==> can ship some products from i to j

...........

var Trans {SHIP} >= 0;   # actual units to be shipped

var Use {LINK} binary;   # 1 if link used, 0 otherwise

minimize Total_Cost:

   sum {(i,j,p) in SHIP} vcost[i,j,p] * Trans[i,j,p]

 + sum {(i,j) in LINK} fcost[i,j] * Use[i,j];
```

Multicommodity Transportation

AMPL “Sparse” Network

Constraint for dense network

```
subject to Supply {i in ORIG, p in PROD}:
   sum {j in DEST} Trans[i,j,p] <= supply[i,p];
```

Constraint for sparse network

```
subject to Supply {i in ORIG, p in PROD}:
   sum {(i,j,p) in SHIP} Trans[i,j,p] <= supply[i,p];
```

Multicommodity Transportation

AMPL “Sparse” Network

All constraints

```
subject to Supply {i in ORIG, p in PROD}:
   sum {(i,j,p) in SHIP} Trans[i,j,p] <= supply[i,p];

subject to Demand {j in DEST, p in PROD}:
   sum {(i,j,p) in SHIP} Trans[i,j,p] = demand[j,p];

subject to Multi {i in ORIG, j in DEST}:
   sum {(i,j,p) in SHIP} Trans[i,j,p] <= limit[i,j] * Use[i,j];

subject to Min_Ship {i in ORIG, j in DEST}:
   sum {(i,j,p) in SHIP} Trans[i,j,p] >= minload * Use[i,j];

subject to Max_Serve {i in ORIG}:
   sum {(i,j) in LINK} Use[i,j] <= maxserve;
```

Multicommodity Transportation

AMPL “Sparse” Network

1st dataset: shipments allowed

```
set SHIP :=

 (*,*,bands):  FRA  DET  LAN  WIN  STL  FRE  LAF :=
       GARY      +    +    +    +    +    -    +
       CLEV      +    -    +    -    +    +    +
       PITT      -    +    +    +    +    +    +

 (*,*,coils):  FRA  DET  LAN  WIN  STL  FRE  LAF :=
       GARY      +    +    +    +    +    +    -
       CLEV      +    +    -    +    +    +    +
       PITT      +    +    +    +    +    +    +

 (*,*,plate):  FRA  DET  LAN  WIN  STL  FRE  LAF :=
       GARY      +    +    -    +    +    -    +
       CLEV      +    +    +    +    +    +    +
       PITT      -    +    +    -    +    +    + ;
```

Multicommodity Transportation

AMPL “Sparse” Network

2nd dataset: shipments allowed

```
set SHIP :=

 (*,*,bands):  FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY     +    +    +    +    +    -    -
        CLEV     -    +    +    -    +    +    +
        PITT     +    -    +    +    +    +    +

 (*,*,coils):  FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY     +    +    +    +    +    +    +
        CLEV     +    +    -    +    +    +    +
        PITT     +    +    +    +    +    +    +

 (*,*,plate):  FRA  DET  LAN  WIN  STL  FRE  LAF :=
        GARY     -    +    +    +    +    -    +
        CLEV     +    +    +    +    +    +    +
        PITT     +    +    -    -    +    +    + ;
```

Multicommodity Transportation

AMPL “Sparse” Network

Same model, different data

```
ampl: model multmipT.mod;
ampl: data multmipT1.dat;

ampl: solve;

Gurobi 4.6.0: optimal solution; objective 247725
108 simplex iterations
13 branch-and-cut nodes

ampl: reset data;
ampl: data multmipT2.dat;

ampl: solve;

Gurobi 4.6.0: optimal solution; objective 237775
79 simplex iterations

ampl:
```

1: Parametric Analysis

Try different limits on destinations served

- Reduce parameter `maxserve` and re-solve
  - until there is no feasible solution
- Display results
  - parameter value
  - numbers of destinations actually served

Try different supplies of plate at Gary

- Increase parameter `supply['GARY','plate']` and re-solve
  - until dual is zero (constraint is slack)
- Record results
  - distinct dual values
  - corresponding objective values

. . . display results at the end

Parametric Analysis *on limits*

Script to test sensitivity to serve limit

```
model multmipG.mod;
data multmipG.dat;

option solver gurobi;

for {m in 7..1 by -1} {
   let maxserve := m;
   solve;
   if solve_result = 'infeasible' then break;
   display maxserve, Max_Serve.body;
}
```

Parametric Analysis *on limits*

Run showing sensitivity to serve limit

```
ampl: include multmipServ.run;

Gurobi 4.6.0: optimal solution; objective 233150
maxserve = 7
CLEV 5   GARY 3   PITT 6

Gurobi 4.6.0: optimal solution; objective 233150
maxserve = 6
CLEV 5   GARY 3   PITT 6

Gurobi 4.6.0: optimal solution; objective 235625
maxserve = 5
CLEV 5   GARY 3   PITT 5

Gurobi 4.6.0: infeasible
```

Parametric Analysis *on supplies*

Script to test sensitivity to plate supply at GARY

```
set SUPPLY default {};
param sup_obj {SUPPLY};
param sup_dual {SUPPLY};

let supply['GARY','plate'] := 200;
param sup_step = 10;
param previous_dual default -Infinity;

repeat while previous_dual < 0 {

  solve;

  if Supply['GARY','plate'].dual > previous_dual then {
    let SUPPLY := SUPPLY union {supply['GARY','plate']};
    let sup_obj[supply['GARY','plate']] := Total_Cost;
    let sup_dual[supply['GARY','plate']] := Supply['GARY','plate'].dual;
    let previous_dual := Supply['GARY','plate'].dual;
    }

  let supply['GARY','plate'] := supply['GARY','plate'] + supply_step;
  }
```

Parametric Analysis *on supplies*

Run showing sensitivity to plate supply at GARY

```
ampl: include multmipSupply.run;

ampl: display sup_obj, sup_dual;
:      sup_obj   sup_dual    :=
200     223504    -13
380     221171    -11.52
460     220260    -10.52
510     219754     -8.52
560     219413      0
;
```

Parametric: Observations

Results of solve can be tested

- Check whether problem is no longer feasible
  - `if solve_result = 'infeasible' then break;`

Parameters are true objects

- Assign new value to param **`supply`**

  -
    ```
    let supply['GARY','plate'] :=
          supply['GARY','plate'] + supply_step;
    ```
- Problem instance changes accordingly

Sets are true data

- Assign new value to set `SUPPLY`
  - `let SUPPLY := SUPPLY union {supply['GARY','plate']};`
- All indexed entities change accordingly

2a: Solution Generation *via Cuts*

Same multicommodity transportation model

Generate n best solutions using different routes

- Display routes used by each solution

Solutions *via Cuts*

Script

```
param nSols default 0;
param maxSols = 3;

model multmipG.mod;
data multmipG.dat;

set USED {1..nSols} within {ORIG,DEST};

subject to exclude {k in 1..nSols}:

   sum {(i,j) in USED[k]} (1-Use[i,j]) +
   sum {(i,j) in {ORIG,DEST} diff USED[k]} Use[i,j] >= 1;

repeat {
   solve;
   display Use;
   let nSols := nSols + 1;
   let USED[nSols] := {i in ORIG, j in DEST: Use[i,j] > .5};
} until nSols = maxSols;
```

AMPL Scripting

Run showing 3 best solutions

```
ampl: include multmipBestA.run;

Gurobi 4.6.0: optimal solution; objective 235625

:    DET FRA FRE LAF LAN STL WIN    :=
CLEV   1   1   1   0   1   1   0
GARY   0   0   0   1   0   1   1
PITT   1   1   1   1   0   1   0 ;

Gurobi 4.6.0: optimal solution; objective 237125

:    DET FRA FRE LAF LAN STL WIN    :=
CLEV   1   1   1   1   0   1   0
GARY   0   0   0   1   0   1   1
PITT   1   1   1   0   1   1   0 ;

Gurobi 4.6.0: optimal solution; objective 238225

:    DET FRA FRE LAF LAN STL WIN    :=

CLEV   1   0   1   0   1   1   1
GARY   0   1   0   1   0   1   0
PITT   1   1   1   1   0   1   0 ;
```

Solutions *via Cuts:* Observations

Same expressions describe sets and indexing

- Index a summation
  - `... sum {(i,j) in {ORIG,DEST} diff USED[k]} Use[i,j] >= 1;`
- Assign a value to a set
  - `let USED[nSols] := {i in ORIG, j in DEST: Use[i,j] > .5};`

New cuts defined automatically

- Index cuts over a set
  - `subject to exclude {k in 1..nSols}: ...`
- Add a cut by expanding the set
  - `let nSols := nSols + 1;`

2b: Solution Generation *via Solver*

Same model

Ask solver to return multiple solutions

- Set options
- Get all results from one “solve”
- Retrieve and display each solution

Solutions *via Solver*

Script

```
option solver cplex;

option cplex_options "poolstub=multmip poolcapacity=3 \
   populate=1 poolintensity=4 poolreplace=1";

solve;

for {i in 1..Current.npool} {

   solution (‘‘multmip" & i & ".sol");
   display Use;

}
```

Solutions *via* Solver

Results

```
ampl: include multmipBestB.run;

CPLEX 12.4.0.0: poolstub=multmip
poolcapacity=3
populate=1
poolintensity=4
poolreplace=1

CPLEX 12.4.0.0: optimal integer solution; objective 235625
439 MIP simplex iterations
40 branch-and-bound nodes

Wrote 3 solutions in solution pool
to files multmip1.sol ... multmip3.sol.

Suffix npool OUT;
```

Solutions *via* Solver

Results (continued)

```
Solution pool member 1 (of 3); objective 235625

:    DET FRA FRE LAF LAN STL WIN  :=
CLEV   1   1   1   0   1   1   0
GARY   0   0   0   1   0   1   1
PITT   1   1   1   1   0   1   0 ;

Solution pool member 2 (of 3); objective 238225

:    DET FRA FRE LAF LAN STL WIN :=
CLEV   1   0   1   0   1   1   1
GARY   0   1   0   1   0   1   0
PITT   1   1   1   1   0   1   0 ;

Solution pool member 3 (of 3); objective 237125

:    DET FRA FRE LAF LAN STL WIN :=
CLEV   1   1   1   1   0   1   0
GARY   0   0   0   1   0   1   1
PITT   1   1   1   0   1   1   0 ;
```

Solutions *via Solver:* Observations

Filenames can be formed dynamically

- Write a (string expression)
- Numbers are automatically converted
  - `solution ("multmip" & i & ".sol");`

3: Heuristic Optimization

Workforce planning

- Cover demands for workers
  - Each “shift” requires a certain number of employees
  - Each employee works a certain “schedule” of shifts
- Satisfy scheduling rules
  - Only “valid” schedules from given list may be used
  - ***Each schedule that is used at all must be worked by at least ?? employees***
- Minimize total workers needed
  - Which schedules should be used?
  - How many employees should work each schedule?

Difficult instances

- Set ?? to a “hard” value
- Get a very good solution quickly

Heuristic

Model (sets, parameters)

```
set SHIFTS;                  # shifts

param Nsched;                # number of schedules;
set SCHEDS = 1..Nsched;      # set of schedules

set SHIFT_LIST {SCHEDS} within SHIFTS;

param rate {SCHEDS} >= 0;       # pay rates
param required {SHIFTS} >= 0;   # staffing requirements

param least_assign >= 0;        # min workers on any schedule used
```

Heuristic

Model (variables, objective, constraints)

```
var Work {SCHEDS} >= 0 integer;
var Use  {SCHEDS} >= 0 binary;

minimize Total_Cost:
   sum {j in SCHEDS} rate[j] * Work[j];

subject to Shift_Needs {i in SHIFTS}:
   sum {j in SCHEDS: i in SHIFT_LIST[j]} Work[j] >= required[i];

subject to Least_Use1 {j in SCHEDS}:
   least_assign * Use[j] <= Work[j];

subject to Least_Use2 {j in SCHEDS}:
   Work[j] <= (max {i in SHIFT_LIST[j]} required[i]) * Use[j];
```

Heuristic

Data

```
set SHIFTS := Mon1 Tue1 Wed1 Thu1 Fri1 Sat1
              Mon2 Tue2 Wed2 Thu2 Fri2 Sat2
              Mon3 Tue3 Wed3 Thu3 Fri3 ;

param Nsched := 126 ;
set SHIFT_LIST[1] := Mon1 Tue1 Wed1 Thu1 Fri1 ;
set SHIFT_LIST[2] := Mon1 Tue1 Wed1 Thu1 Fri2 ;
set SHIFT_LIST[3] := Mon1 Tue1 Wed1 Thu1 Fri3 ;
set SHIFT_LIST[4] := Mon1 Tue1 Wed1 Thu1 Sat1 ;
set SHIFT_LIST[5] := Mon1 Tue1 Wed1 Thu1 Sat2 ;  .......

param required :=  Mon1 100  Mon2 78  Mon3 52
                   Tue1 100  Tue2 78  Tue3 52
                   Wed1 100  Wed2 78  Wed3 52
                   Thu1 100  Thu2 78  Thu3 52
                   Fri1 100  Fri2 78  Fri3 52
                   Sat1 100  Sat2 78 ;
```

Heuristic

Hard case: `least_assign = 19`

```
ampl: model sched1.mod;
ampl: data sched.dat;

ampl: let least_assign := 19;

ampl: option solver cplex;
ampl: solve;

CPLEX 12.2.0.2: optimal integer solution; objective 269
635574195 MIP simplex iterations
86400919 branch-and-bound nodes

ampl: option omit_zero_rows 1, display_1col 0;
ampl: display Work;

Work [*] :=
  4 22    16 39    55 39    78 39    101 39    106 52    122 39
;
```

. . . ***94.8 minutes***

Heuristic

Alternative, indirect approach

- Step 1: Relax integrality of `Work` variables
  Solve for zero-one `Use` variables
- Step 2: Fix `Use` variables
  Solve for integer `Work` variables

. . . ***not necessarily optimal, but*** . . .

Heuristic

Script

```
model sched1.mod;
data sched.dat;

let least_assign := 19;

let {j in SCHEDS} Work[j].relax := 1;

solve;

fix {j in SCHEDS} Use[j];
let {j in SCHEDS} Work[j].relax := 0;

solve;
```

Heuristic

Results

```
ampl: include sched1-fix.run;

CPLEX 12.2.0.2: optimal integer solution; objective 268.5
32630436 MIP simplex iterations
2199508 branch-and-bound nodes

Work [*] :=
  1 24      32 19      80 19.5   107 33     126 19.5
  3 19      66 19      90 19.5   109 19
 10 19      72 19.5   105 19.5   121 19 ;

CPLEX 12.2.0.2: optimal integer solution; objective 269
2 MIP simplex iterations
0 branch-and-bound nodes

Work [*] :=
  1 24    10 19    66 19    80 19    105 20    109 19    126 20
  3 19    32 19    72 19    90 20    107 33    121 19 ;
```

. . . ***2.85 minutes***

Heuristic: Observations

Models can be changed dynamically

- Adapt modeling expressions
- Execute model-related commands
  - `fix {j in SCHEDS} Use[j];`
- Assign values to properites of model components
  - `let {j in SCHEDS} Work[j].relax := 1;`

4: Pattern Generation

Roll cutting

- Min rolls cut (or material wasted)
- Decide number of each pattern to cut
- Meet demands for each ordered width

Generate cutting patterns

- Read general model
- Read data: demands, raw width
- Compute data: all usable patterns
- Solve problem instance

Pattern Generation

Model

```
param roll_width > 0;

set WIDTHS ordered by reversed Reals;
param orders {WIDTHS} > 0;

param maxPAT integer >= 0;
param nPAT integer >= 0, <= maxPAT;

param nbr {WIDTHS,1..maxPAT} integer >= 0;

var Cut {1..nPAT} integer >= 0;

minimize Number:

   sum {j in 1..nPAT} Cut[j];

subj to Fulfill {i in WIDTHS}:

   sum {j in 1..nPAT} nbr[i,j] * Cut[j] >= orders[i];
```

Pattern Generation

Data

```
param roll_width := 90 ;

param: WIDTHS: orders :=

        60        3
        30       21
        25.5     94
        20       50
        17.25   288
        15      178
        12.75   112
        10      144  ;
```

Pattern Generation

Script (initialize)

```
model cutPAT.mod;
data ChvatalD.dat;

model;
param curr_sum >= 0;
param curr_width > 0;
param pattern {WIDTHS} integer >= 0;

let maxPAT := 100000000;

let nPAT := 0;
let curr_sum := 0;
let curr_width := first(WIDTHS);
let {w in WIDTHS} pattern[w] := 0;
```

Pattern Generation

Script (loop)

```
repeat {
   if curr_sum + curr_width <= roll_width then {
      let pattern[curr_width] := floor((roll_width-curr_sum)/curr_width);
      let curr_sum := curr_sum + pattern[curr_width] * curr_width;
      }
   if curr_width != last(WIDTHS) then
      let curr_width := next(curr_width,WIDTHS);
   else {
      let nPAT := nPAT + 1;
      let {w in WIDTHS} nbr[w,nPAT] := pattern[w];
      let curr_sum := curr_sum - pattern[last(WIDTHS)] * last(WIDTHS);
      let pattern[last(WIDTHS)] := 0;
      let curr_width := min {w in WIDTHS: pattern[w] > 0} w;
      if curr_width < Infinity then {
         let curr_sum := curr_sum - curr_width;
         let pattern[curr_width] := pattern[curr_width] - 1;
         let curr_width := next(curr_width,WIDTHS);
         }
      else break;
      }
   }
```

Pattern Generation

Script (solve, report)

```
option solver gurobi;

solve;

printf "\n%5i patterns, %3i rolls", nPAT, sum {j in 1..nPAT} Cut[j];
printf "\n\n  Cut   ";
printf {j in 1..nPAT: Cut[j] > 0}: "%3i", Cut[j];
printf "\n\n";

for {i in WIDTHS} {
   printf "%7.2f ", i;
   printf {j in 1..nPAT: Cut[j] > 0}: "%3i", nbr[i,j];
   printf "\n";
   }

printf "\nWASTE = %5.2f%%\n\n",
   100 * (1 - (sum {i in WIDTHS} i * orders[i]) / (roll_width * Number));
```

Pattern Generation

Results

```
ampl: include cutPatEnum.run

Gurobi 4.6.1: optimal solution; objective 164
15 simplex iterations

  290 patterns, 164 rolls

  Cut      3  7 50 44 17 25  2 16

  60.00    1  0  0  0  0  0  0  0
  30.00    0  3  0  0  0  0  0  0
  25.50    0  0  1  1  0  0  0  0
  20.00    0  0  0  0  3  0  0  0
  17.25    0  0  3  2  0  2  0  0
  15.00    2  0  0  2  2  2  0  0
  12.75    0  0  1  0  0  2  7  0
  10.00    0  0  0  0  0  0  0  9

WASTE =  0.32%
```

Pattern Generation

Data 2

```
param roll_width := 349 ;

param: WIDTHS: orders :=
       28.75     7
       33.75    23
       34.75    23
       37.75    31
       38.75    10
       39.75    39
       40.75    58
       41.75    47
       42.25    19
       44.75    13
       45.75    26 ;
```

Pattern Generation

Results 2

```
ampl: include cutPatEnum.run

Gurobi 4.6.1: optimal solution; objective 34
291 simplex iterations

54508 patterns,  34 rolls

  Cut      8  1  1  1  3  1  1  1  1  2  7  2  3  1  1

  45.75    3  2  0  0  0  0  0  0  0  0  0  0  0  0  0
  44.75    1  2  2  1  0  0  0  0  0  0  0  0  0  0  0
  42.25    0  2  0  0  4  2  2  1  0  0  0  0  0  0  0
  41.75    4  2  0  2  0  0  0  0  2  1  1  0  0  0  0
  40.75    0  0  4  4  1  4  3  0  2  3  1  6  3  2  2
  39.75    0  0  0  0  0  0  0  2  0  0  5  0  0  2  0
  38.75    0  0  1  0  0  0  0  0  4  0  0  0  0  2  3
  37.75    0  0  0  0  0  0  1  0  0  4  0  0  6  2  4
  34.75    0  0  0  0  4  0  3  1  0  0  0  3  0  1  0
  33.75    0  0  0  0  0  3  0  4  0  1  2  0  0  0  0
  28.75    0  0  2  2  0  0  0  2  1  0  0  0  0  0  0

WASTE =  0.69%
```

Pattern Generation

Data 3

```
param roll_width := 172 ;

param: WIDTHS: orders :=
       25.000     5
       24.750    73
       18.000    14
       17.500     4
       15.500    23
       15.375     5
       13.875    29
       12.500    87
       12.250     9
       12.000    31
       10.250     6
       10.125    14
       10.000    43
        8.750    15
        8.500    21
        7.750     5 ;
```

Pattern Generation

Results 3 (using a subset of patterns)

```
ampl: include cutPatEnum.run

Gurobi 4.6.1: optimal solution; objective 33
722 simplex iterations
40 branch-and-cut nodes

273380 patterns,  33 rolls

  Cut     1  1  1  1  4  4  4  1  1  2  5  2  1  1  1  3

  25.00   2  1  1  1  0  0  0  0  0  0  0  0  0  0  0  0
  24.75   1  2  1  0  5  4  3  2  2  2  2  1  1  0  0  0
  18.00   0  0  0  0  1  0  0  1  0  0  0  1  1  5  1  0
  17.50   0  3  0  0  0  0  0  0  0  0  0  0  0  0  1  0
  .......
  10.12   0  2  0  0  0  1  2  0  0  0  0  0  0  0  0  0
  10.00   0  0  0  0  0  2  0  1  3  0  6  0  0  2  0  0
   8.75   0  0  1  0  0  0  0  0  0  2  0  2  0  0  0  2
   8.50   0  0  2  0  0  2  0  0  0  0  0  4  3  0  0  0
   7.75   0  0  0  0  1  0  0  1  0  0  0  0  0  0  0  0

WASTE =  0.62%
```

Pattern Generation: Observations

Parameters can serve as script variables

- Declare as in model
  - `param pattern {WIDTHS} integer >= 0;`
- Use in algorithm
  - `let pattern[curr_width] := pattern[curr_width] - 1;`
- Assign to model parameters
  - `let {w in WIDTHS} nbr[w,nPAT] := pattern[w];`

Scripts are easy to modify

- Store only every 100th pattern found
  -
    ```
    if nPAT mod 100 = 0 then
       let {w in WIDTHS} nbr[w,nPAT/100] := pattern[w];
    ```

5: Decomposition

Stochastic nonlinear location-transportation

- Min expected total cost
  - Nonlinear construction costs at origins
  - Linear transportation costs from origins to destinations
- Stochastic demands with recourse
  - Decide what to build
  - Observe demands and decide what to ship

Solve by Benders decomposition

- Nonlinear master problem
- Linear subproblem for each scenario

Decomposition

Original model (sets, parameters, variables)

```
set WHSE;   # shipment origins (warehouses)
set STOR;   # shipment destinations (stores)

param build_cost {WHSE} > 0;    # costs per unit to build warehouse
param build_limit {WHSE} > 0;   # limits on units shipped

var Build {i in WHSE} >= 0, <= .9999 * build_limit[i];
                                # capacities of warehouses to build

set SCEN;                       # demand scenarios

param prob {SCEN} >= 0, <= 1;   # probabilities of scenarios
param demand {STOR,SCEN} >= 0;  # amounts required at stores

param ship_cost {WHSE,STOR} >= 0;  # shipment costs per unit

var Ship {WHSE,STOR,SCEN} >= 0;    # amounts to be shipped
```

Decomposition

Original model (objective, constraints)

```
minimize Total_Cost:

   sum {i in WHSE}
      build_cost[i] * Build[i] / (1 - Build[i]/build_limit[i]) +

   sum {s in SCEN} prob[s] *
      sum {i in WHSE, j in STOR} ship_cost[i,j] * Ship[i,j,s];

subj to Supply {i in WHSE, s in SCEN}:

   sum {j in STOR} Ship[i,j,s] <= Build[i];

subj to Demand {j in STOR, s in SCEN}:

   sum {i in WHSE} Ship[i,j,s] = demand[j,s];
```

Decomposition

Sub model (sets, parameters, variables)

```
set WHSE;   # shipment origins (warehouses)
set STOR;   # shipment destinations (stores)
param build {i in WHSE} >= 0, <= .9999 * build_limit[i];
                                # capacities of warehouses built
set SCEN;                       # demand scenarios
param prob {SCEN} >= 0, <= 1;   # probabilities of scenarios
param demand {STOR,SCEN} >= 0;  # amounts required at stores

param ship_cost {WHSE,STOR} >= 0;  # shipment costs per unit
var Ship {WHSE,STOR,SCEN} >= 0;    # amounts to be shipped
```

Decomposition

Sub model (objective, constraints)

```
param S symbolic in SCEN;

minimize Scen_Ship_Cost:
   prob[S] * sum {i in WHSE, j in STOR} ship_cost[i,j] * Ship[i,j];

subj to Supply {i in WHSE}:
   sum {j in STOR} Ship[i,j] <= build[i];

subj to Demand {j in STOR}:
   sum {i in WHSE} Ship[i,j] = demand[j,S];
```

Decomposition

Master model (sets, parameters, variables)

```
param build_cost {WHSE} > 0;    # costs per unit to build warehouse
param build_limit {WHSE} > 0;   # limits on units shipped

var Build {i in WHSE} >= 0, <= .9999 * build_limit[i];
                                # capacities of warehouses to build

param nCUT >= 0 integer;

param cut_type {SCEN,1..nCUT} symbolic
   within {"feas","infeas","none"};

param supply_price {WHSE,SCEN,1..nCUT} <= 0.000001;
param demand_price {STOR,SCEN,1..nCUT};

var Max_Exp_Ship_Cost {SCEN} >= 0;
```

Decomposition

Master model (objective, constraints)

```
minimize Expected_Total_Cost:
   sum {i in WHSE}
      build_cost[i] * Build[i] / (1 - Build[i]/build_limit[i]) +
   sum {s in SCEN} Max_Exp_Ship_Cost[s];

subj to Cut_Defn {s in SCEN, k in 1..nCUT: cut_type[s,k] != "none"}:
   if cut_type[s,k] = "feas" then Max_Exp_Ship_Cost[s] else 0 >=
      sum {i in WHSE} supply_price[i,s,k] * Build[i] +
      sum {j in STOR} demand_price[j,s,k] * demand[j,s];
```

Decomposition

Script (initialization)

```
model stbenders.mod;
data stnltrnloc.dat;

suffix dunbdd;
option presolve 0;

problem Sub: Ship, Scen_Ship_Cost, Supply, Demand;
   option solver cplex;
   option cplex_options 'primal presolve 0';

problem Master: Build, Max_Exp_Ship_Cost, Exp_Total_Cost, Cut_Defn;
   option solver minos;

let nCUT := 0;

param GAP default Infinity;
param RELGAP default Infinity;
param Exp_Ship_Cost;
```

Decomposition

Script (iteration)

```
repeat {

   solve Master;

   let {i in WHSE} build[i] := Build[i];
   let Exp_Ship_Cost := 0;
   let nCUT := nCUT + 1;

   for {s in SCEN} {

      let S := s;
      solve Sub;

      ... generate a cut ...
      }

   if forall {s in SCEN} cut_type[s,nCUT] != "infeas" then {
      let GAP := min (GAP,
         Exp_Ship_Cost - sum {s in SCEN} Max_Exp_Ship_Cost[s]);
      let RELGAP := 100 * GAP / Expected_Total_Cost;
      }

} until RELGAP <= .000001;
```

Decomposition

Script (cut generation)

```
for {s in SCEN} {
   let S := s;
   solve Sub;

   if Sub.result = "solved" then {
      let Exp_Ship_Cost := Exp_Ship_Cost + Scen_Ship_Cost;

      if Scen_Ship_Cost > Max_Exp_Ship_Cost[s] + 0.00001 then {
         let cut_type[s,nCUT] := "feas";
         let {i in WHSE} supply_price[i,s,nCUT] := Supply[i].dual;
         let {j in STOR} demand_price[j,s,nCUT] := Demand[j].dual;
         }

      else let cut_type[s,nCUT] := "none";
      }

   else if Sub.result = "infeasible" then {
      let cut_type[s,nCUT] := "infeas";
      let {i in WHSE} supply_price[i,s,nCUT] := Supply[i].dunbdd;
      let {j in STOR} demand_price[j,s,nCUT] := Demand[j].dunbdd;
      }
   }
```

Decomposition

Results

```
ampl: include stbenders.run;

MASTER PROBLEM 1: 0.000000

SUB-PROBLEM 1  low: infeasible
SUB-PROBLEM 1  mid: infeasible
SUB-PROBLEM 1 high: infeasible

MASTER PROBLEM 2: 267806.267806

SUB-PROBLEM 2  low: 1235839.514234
SUB-PROBLEM 2  mid: 1030969.048921
SUB-PROBLEM 2 high: infeasible

MASTER PROBLEM 3: 718918.236014

SUB-PROBLEM 3  low: 1019699.661119
SUB-PROBLEM 3  mid: 802846.293052
SUB-PROBLEM 3 high: 695402.974379

GAP = 2517948.928551, RELGAP = 350.241349%
```

Decomposition

Results (continued)

```
MASTER PROBLEM 4: 2606868.719958

SUB-PROBLEM 4  low: 1044931.784272
SUB-PROBLEM 4  mid: 885980.640150
SUB-PROBLEM 4 high: 944581.118758

GAP = 749765.716399, RELGAP = 28.761161%

MASTER PROBLEM 5: 2685773.838398

SUB-PROBLEM 5  low: 1028785.052062
SUB-PROBLEM 5  mid: 815428.531237
SUB-PROBLEM 5 high: 753627.189086

GAP = 394642.837091, RELGAP = 14.693822%

MASTER PROBLEM 6: 2743483.001029

SUB-PROBLEM 6  low: 1000336.408156
SUB-PROBLEM 6  mid: 785602.983289
SUB-PROBLEM 6 high: 725635.817601

GAP = 222288.965560, RELGAP = 8.102436%
```

Decomposition

Results (continued)

```
MASTER PROBLEM 7: 2776187.713412

SUB-PROBLEM 7  low: 986337.500000
SUB-PROBLEM 7  mid: 777708.466300
SUB-PROBLEM 7 high: 693342.659287

GAP = 59240.084058, RELGAP = 2.133864%

MASTER PROBLEM 8: 2799319.395374

SUB-PROBLEM 8  low: 991426.284976
SUB-PROBLEM 8  mid: 777146.351060
SUB-PROBLEM 8 high: 704353.854398

GAP = 38198.286498, RELGAP = 1.364556%

MASTER PROBLEM 9: 2814772.778136

SUB-PROBLEM 9  low: 987556.309573
SUB-PROBLEM 9  mid: 772147.258329
SUB-PROBLEM 9 high: 696060.666966

GAP = 17658.226624, RELGAP = 0.627341%
```

Decomposition

Results (continued)

```
MASTER PROBLEM 10: 2818991.649514

SUB-PROBLEM 10  mid: 771853.500000
SUB-PROBLEM 10 high: 689709.131427

GAP = 2361.940101, RELGAP = 0.083787%

MASTER PROBLEM 11: 2819338.502316

SUB-PROBLEM 11 high: 692406.351318

GAP = 2361.940101, RELGAP = 0.083776%

MASTER PROBLEM 12: 2819524.204253

SUB-PROBLEM 12 high: 690478.286312

GAP = 541.528304, RELGAP = 0.019206%

MASTER PROBLEM 13: 2819736.994159

GAP = -0.000000, RELGAP = -0.000000%

OPTIMAL SOLUTION FOUND
Expected Cost = 2819736.994159
```

Decomposition: Observations

Loops can iterate over sets

- ❖ Solve a subproblem for each scenario
  - `for {s in SCEN} { ...`

One model can represent all subproblems

- ❖ Assign loop index `s` to set `S`, then solve

  -
    ```
    let S := s;
    solve Sub;
    ```

Related solution values can be returned

- ❖ Use dual ray to generate infeasibility cuts

  -
    ```
    if Sub.result = "infeasible" then { ...
      let {i in WHSE}
        supply_price[i,s,nCUT] := Supply[i].dunbdd;
      let {j in STOR}
        demand_price[j,s,nCUT] := Demand[j].dunbdd;
      }
    ```

Concluding Observations

Scripts in practice

- Large and complicated
  - Multiple files
  - Hundreds of statements
  - Millions of statements executed
- Run within broader applications

Prospective improvements

- Faster loops
- True script functions
  - Arguments and return values
  - Local sets & parameters
- More database connections
- IDE for debugging
- APIs for popular languages (C++, Java, C#, VB, *Python*)