The Origins of a Practical Simplex Method

Robert Fourer

4er@ampl.com

AMPL Optimization Inc.
www.ampl.com — +1 773-336-AMPL

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Session MB08, *Celebrating the Contributions of George Dantzig: Applications and Software*

1948

Programming of Interdependent Activities II: Mathematical Model

- George B. Dantzig
- *Econometrica* **17** (1949)

"Linear Programming"

- Formulations & applications
- No algorithm

"It is proposed to solve linear programming problems . . . by means of large scale digital computers . . . . Several computational procedures have been evolved so far and research is continuing actively in this field."

PROGRAMMING OF INTERDEPENDENT ACTIVITIES
II MATHEMATICAL MODEL[1]

BY GEORGE B. DANTZIG

Activities (or production processes) are considered as building blocks out of which a technology is constructed. Postulates are developed by which activities may be combined. The main part of the paper is concerned with the discrete type model and the use of a linear maximization function for finding the "optimum" program. The mathematical problem associated with this approach is developed first in general notation and then in terms of a dynamic system of equations expressed in matrix notation. Typical problems from the fields of inter-industry relations, transportation, nutrition, warehouse storage, and air transport are given in the last section.

INTRODUCTION

THE MULTITUDE of activities in which a large organization or a nation engages can be viewed not only as fixed objects but as representative building blocks of different kinds that might be recombined in varying amounts to form new blocks. If a structure can be reared of these blocks that is mutually self-supporting, the resulting edifice can be thought of as a technology. Usually the very elementary blocks have a wide variety of forms and quite irregular characteristics over time. Often they are combined with other blocks so that they will have "nicer" characteristics when used to build a complete system. Thus the science of programming, if it may be called a science, is concerned with the adjustment of the levels of a set of given activities (production processes) so that they remain mutually consistent and satisfy certain optimum properties.

It is highly desirable to have formal rules by which activities can be combined to form composite activities and an economy. These rules are set forth here as a set of postulates regarding reality. Naturally other postulates are possible; those selected have been chosen with a wide class of applications in mind and with regard to the limitations of present day computational techniques. The reader's attention is drawn to the last section of this report where a number of applications of the mathematical model are discussed. These are believed to be of sufficient interest in themselves, and may lend concreteness to the development which follows:

POSTULATES OF A LINEAR TECHNOLOGY

POSTULATE I: *There exists a set $\{A\}$ of activities.*
POSTULATE II: *All activities take place within a time span 0 to t_0.*

[1] A revision of a paper presented before the Madison Meeting of the Econometric Society on September 9, 1948. This is the second of two papers on this subject, both appearing in this issue. The first paper, with sub-title "General Discussion," will be referred to by Roman numeral I.

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1947

Maximization of a Linear Function of Variables Subject to Linear Inequalities

- George B. Dantzig
- *Activity Analysis of Production and Allocation* (1951)

"Simplex Method"

- Proof of convergence
- No computers

> "As a practical computing matter the iterative procedure of shifting from one basis to the next is not as laborious as would first appear . . ."

Chapter XXI

MAXIMIZATION OF A LINEAR FUNCTION OF VARIABLES SUBJECT TO LINEAR INEQUALITIES[1]

By George B. Dantzig

The general problem indicated in the title is easily transformed, by any one of several methods, to one which maximizes a linear form of nonnegative variables subject to a system of linear equalities. For example, consider the linear inequality $ax + by + c > 0$. The linear inequality can be replaced by a linear equality in nonnegative variables by writing, instead, $a(x_1 - x_2) + b(y_1 - y_2) + c - z = 0$, where $x_1 \geqq 0$, $x_2 \geqq 0$, $y_1 \geqq 0$, $y_2 \geqq 0$, $z \geqq 0$. The basic problem throughout this chapter will be considered in the following form:

Problem: *Find the values of* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *which maximize the linear form*

$$\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n \tag{1}$$

subject to the conditions that

$$\lambda_j \geqq 0 \qquad (j = 1, 2, \cdots, n) \tag{2}$$

and

$$\begin{aligned} \lambda_1 a_{11} + \lambda_2 a_{12} + \cdots + \lambda_n a_{1n} &= b_1, \\ \lambda_1 a_{21} + \lambda_2 a_{22} + \cdots + \lambda_n a_{2n} &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1 a_{m1} + \lambda_2 a_{m2} + \cdots + \lambda_n a_{mn} &= b_m, \end{aligned} \tag{3}$$

where a_{ij}, b_i, c_j *are constants* $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

[1] The author wishes to acknowledge that his work on this subject stemmed from discussions in the spring of 1947 with Marshall K. Wood, in connection with Air Force programming methods. The general nature of the "simplex" approach (as the method discussed here is known) was stimulated by discussions with Leonid Hurwicz.

The author is indebted to T. C. Koopmans, whose constructive observations regarding properties of the simplex led directly to a proof of the method in the early fall of 1947. Emil D. Schell assisted in the preparation of various versions of this chapter. Jack Laderman has written a set of detailed working instructions and has tested this and other proposed techniques on several examples.

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1953

An Introduction to Linear Programming

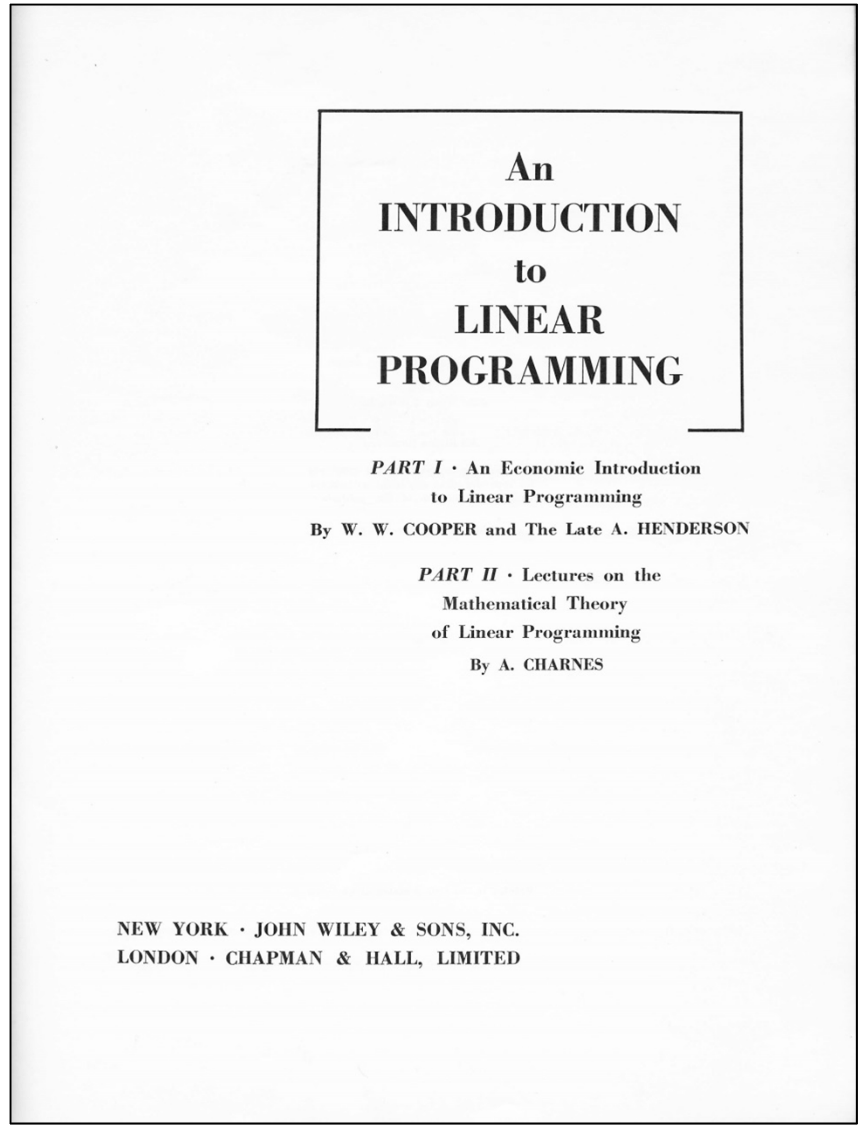
- W.W. Cooper, A. Henderson
- A. Charnes

"Simplex Tableau"

- Symbolic description
- Numerical example

"As far as computations are concerned it is most convenient to arrange the data at each stage in a 'simplex tableau' as shown in Table I.[12]"

"[12]A. Orden suggested this efficient arrangement developed by himself, Dantzig, and Hoffman."



An
INTRODUCTION
to
LINEAR
PROGRAMMING

PART I · An Economic Introduction
to Linear Programming
By W. W. COOPER and The Late A. HENDERSON

PART II · Lectures on the
Mathematical Theory
of Linear Programming
By A. CHARNES

NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

Terminology

Linear program

Minimize $\boldsymbol{c} \cdot \boldsymbol{x}$
Subject to $A\,\boldsymbol{x} = \boldsymbol{b}$
$\boldsymbol{x} \geq \boldsymbol{0}$

Data

$\boldsymbol{b} = (b_1, \ldots, b_m)$
$\boldsymbol{c} = (c_1, \ldots, c_n)$
$A = [a_{ij}]$, with m rows $\boldsymbol{a^i}$ and n columns $\boldsymbol{a_j}$

Variables

$\boldsymbol{x} = (x_1, \ldots, x_n)$

Terminology *(cont'd)*

Basis

- $\mathcal{B}$, $\mathcal{N}$, sets of basic and nonbasic column indices
  - $|\mathcal{B}| = m$, $|\mathcal{N}| = n - m$
- $\boldsymbol{c}_{\mathcal{B}}$, $\boldsymbol{x}_{\mathcal{B}}$, corresponding subvectors of $\boldsymbol{c}$, $\boldsymbol{x}$

Basis matrix

- B, nonsingular $|\mathcal{B}| \times |\mathcal{B}|$ submatrix of A
- $B^{-1} = [z_{ij}]$, with $|\mathcal{B}|$ rows $\boldsymbol{z^i}$ and $|\mathcal{B}|$ columns $\boldsymbol{z_j}$

Tableau Simplex Method

Given

$x_i,\ i \in \mathcal{B}$ (the basic solution)

$y_{ij},\ i \in \mathcal{B},\ j \in \mathcal{N}$ (the tableau)

$d_j,\ j \in \mathcal{N}$ (the reduced costs)

Choose an entering variable

$p \in \mathcal{N}\colon\ d_p < 0$

Choose a leaving variable

$q \in \mathcal{B}\colon\ x_q / y_{qp} = \min_{y_{ip} > 0} x_i / y_{ip}$

Update

- one pivot on a $(|\mathcal{B}| + 1) \times (|\mathcal{N}| + 1)$ tableau
- $x_i \equiv y_{i0},\ d_j \equiv y_{0j}$

Disadvantages

Inefficiency

- ❖ $O(|\mathcal{B}| \times |\mathcal{N}|)$ numbers to compute
- ❖ $O(|\mathcal{B}| \times |\mathcal{N}|)$ numbers to write and store

Instability

- ❖ Rigid computational rules

1953

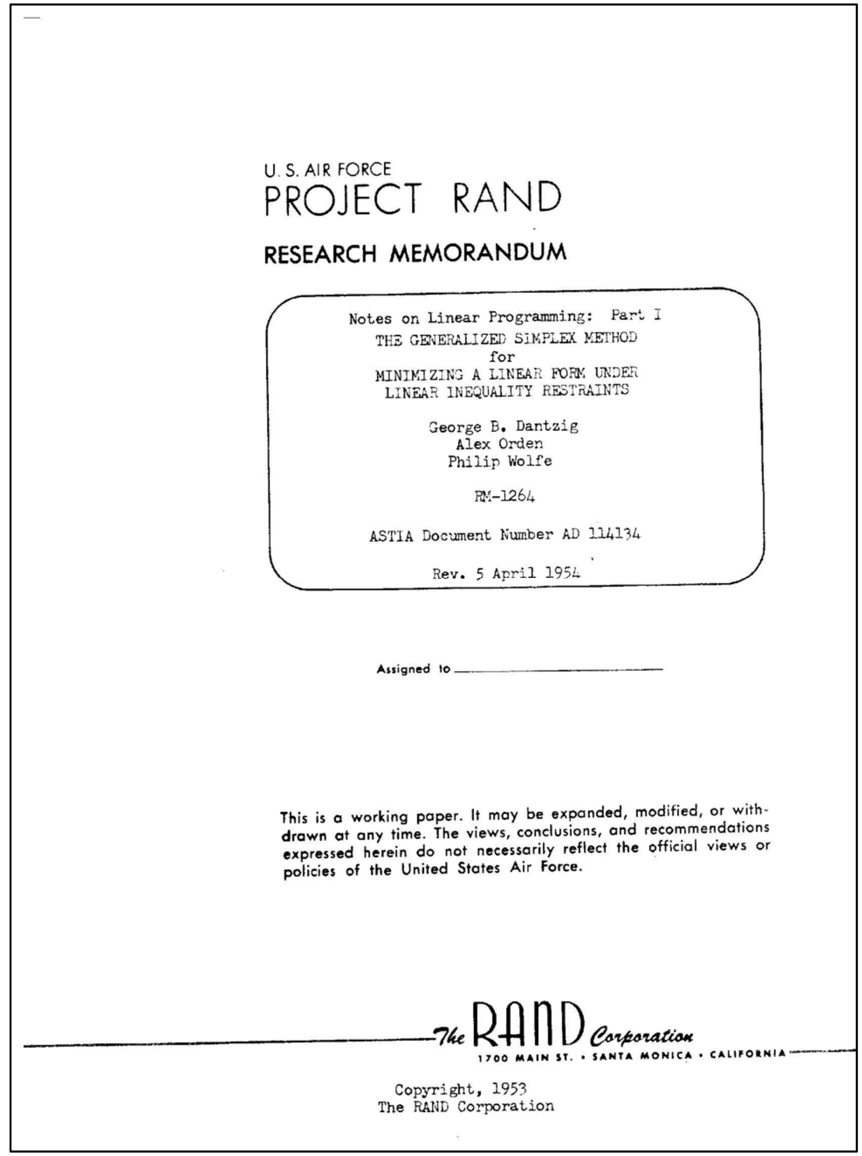
The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints

- George B. Dantzig, Alex Orden, Philip Wolfe
- *Project RAND Research Memorandum RM-1264*

“Lexicographic Simplex Method”

- Prevent degenerate cycling
- Reorganize computations

> “The k+1st iterate is closely related to the kth by simple transformations that constitute the computational algorithm [6], . . .”



U. S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

Notes on Linear Programming: Part I
THE GENERALIZED SIMPLEX METHOD
for
MINIMIZING A LINEAR FORM UNDER
LINEAR INEQUALITY RESTRAINTS

George B. Dantzig
Alex Orden
Philip Wolfe

RM-1264

ASTIA Document Number AD 114134

Rev. 5 April 1954

Assigned to ____________

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
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1951

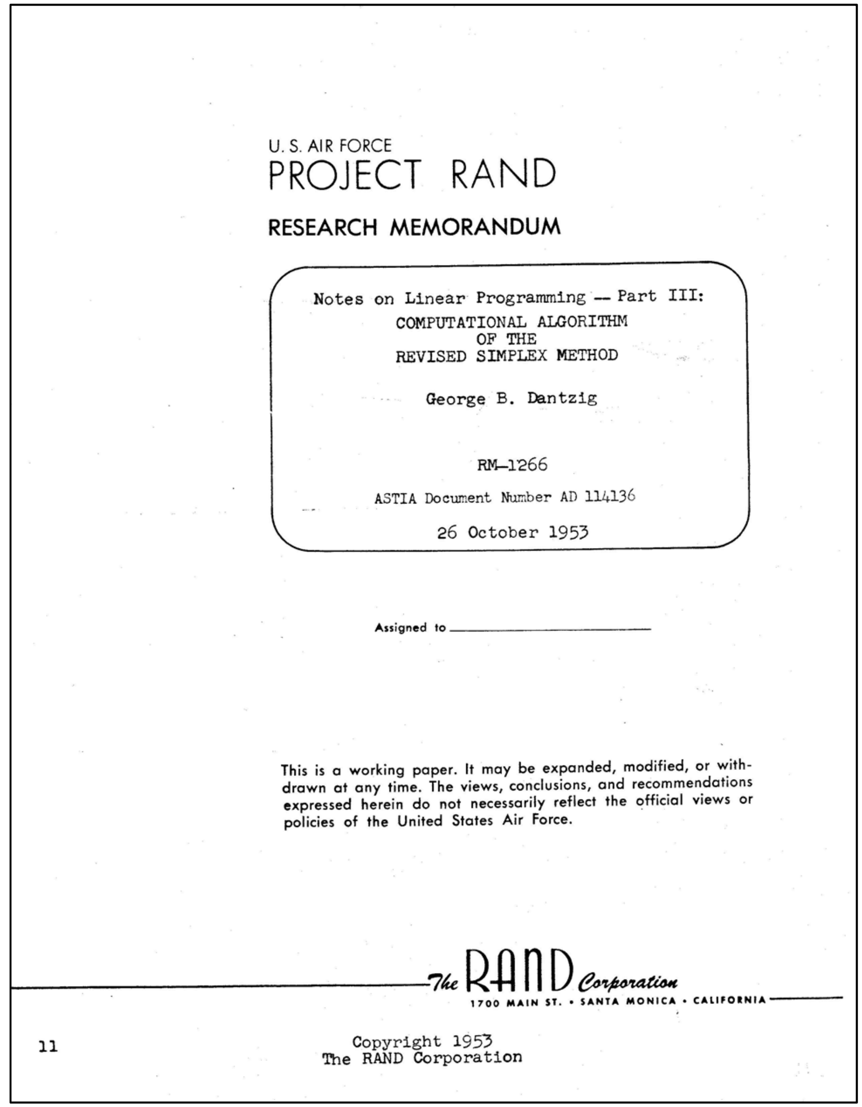
Computational Algorithm of the Revised Simplex Method

- George B. Dantzig
- *Project RAND Research Memorandum RM-1266*

"Revised Simplex Method"

- Break ties for leaving variable
- Update basis inverse

> "The transformation of just the inverse (rather than the entire matrix of coefficients with each cycle) has been developed because it has several important advantages over the old method: . . ."



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PROJECT RAND
RESEARCH MEMORANDUM

Notes on Linear Programming — Part III:
COMPUTATIONAL ALGORITHM
OF THE
REVISED SIMPLEX METHOD

George B. Dantzig

RM-1266

ASTIA Document Number AD 114136

26 October 1953

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Revised Simplex Method

Given

$x_i,\ i \in \mathcal{B}$ (the basic solution)

$z_{ij},\ i \in \mathcal{B},\ j \in \mathcal{B}$ (the basis inverse)

$\pi_i,\ i \in \mathcal{B}$ (the prices)

Choose an entering variable

$p \in \mathcal{N}\colon\ d_p = c_p - \boldsymbol{\pi} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

$y_{ip} = \boldsymbol{z^i} \cdot \boldsymbol{a_p}$

$q \in B\colon\ x_q / y_{qp} = \min_{y_{ip} > 0} x_i / y_{ip}$

Update

- one pivot on a $(|\mathcal{B}| + 1) \times (|\mathcal{B}| + 1)$ tableau
- $x_i \equiv z_{i0},\ \pi_i \equiv z_{0i}$

Advantages

Smaller tableau update

"... In the original method (roughly) $m \times n$ new elements have to be recorded each time. In contrast, the revised method (by making extensive use of cumulative sums of products) requires the recording of about m^2 elements . . . ."

Sparse operations

"In most practical problems the original matrix of coefficients is largely composed of zero elements. . . . The revised method works with the matrix in its original form and takes direct advantage of these zeros."

Disadvantages

Inefficiency

- $|\mathcal{B}| \times |\mathcal{B}|$ numbers to compute
- $|\mathcal{B}| \times |\mathcal{B}|$ numbers to write and store

Instability

- Rigid computational rules

However . . .

> ". . . the revised method (by making extensive use of cumulative sums of products) requires the recording of about m^2 elements (and an alternative method [5] can reduce this to m . . .)."

1953

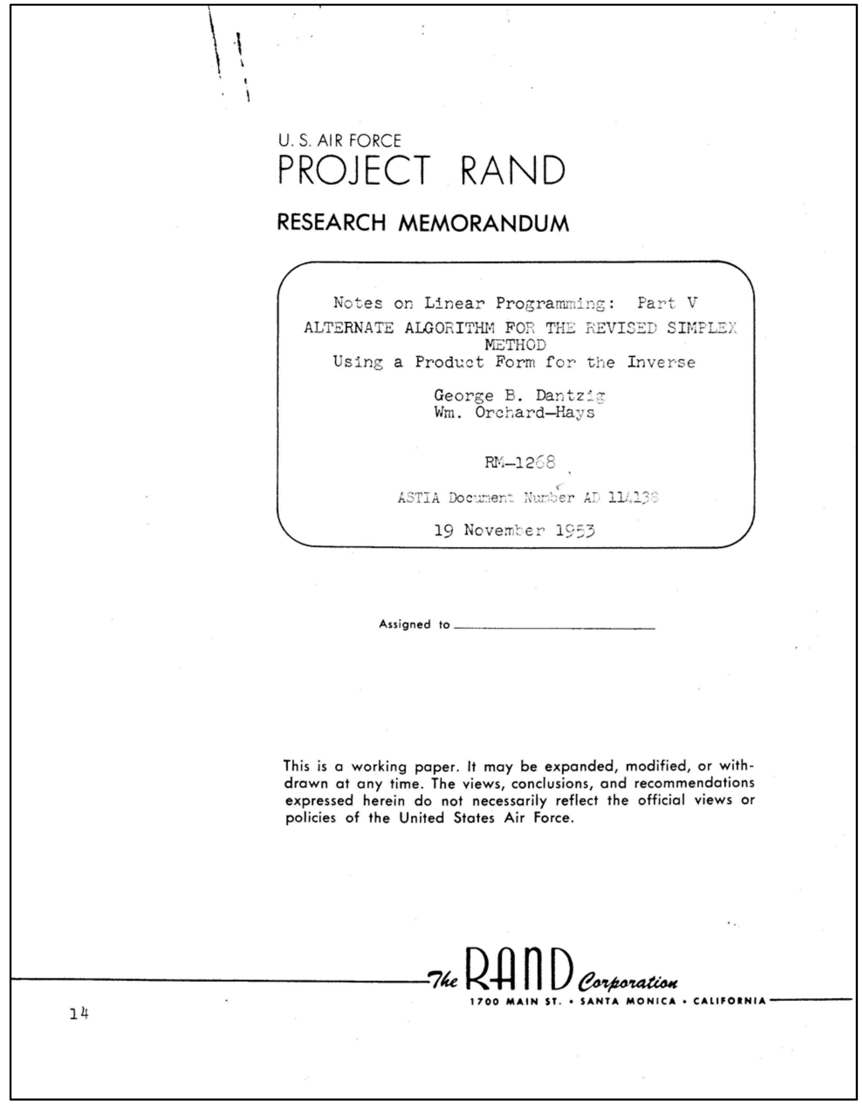
Alternate Algorithm for the Revised Simplex Method

- George B. Dantzig, Wm. Orchard-Hays
- *Project RAND Research Memorandum RM-1268*

"Product Form for the Inverse"

- Fully sparse representation
- Practical computation

> "Using the I.B.M. Card Programmed Calculator, . . . where the inverse matrix is needed at one stage and its transpose at another, this is achieved simply by turning over the deck of cards representing the inverse."



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PROJECT RAND
RESEARCH MEMORANDUM

Notes on Linear Programming: Part V
ALTERNATE ALGORITHM FOR THE REVISED SIMPLEX METHOD
Using a Product Form for the Inverse

George B. Dantzig
Wm. Orchard-Hays

RM-1268

ASTIA Document Number AD 114138

19 November 1953

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Product-Form Simplex Method

Given

$\boldsymbol{x}_{\mathcal{B}}$ (the basic solution)

$B^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$ (factorization of the basis inverse)

Choose an entering variable

$\boldsymbol{\pi} = \boldsymbol{c}_{\mathcal{B}} E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$

$p \in \mathcal{N}\colon\ c_p - \boldsymbol{\pi} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

$\boldsymbol{y_p} = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1} \boldsymbol{a_p}$

$q \in \mathcal{B}\colon\ x_q / y_{qp} = \min_{y_{ip} > 0} x_i / y_{ip}$

Update

- add a factor E_{k+1}^{-1} to the product
- update basic solution to $\boldsymbol{x}_{\mathcal{B}} - (x_q / y_{qp})\, \boldsymbol{y_p}$

Factorization of the Inverse

Form of the factors

- E_i is an identity matrix except for one column
- . . . and so is E_i^{-1}

Storage of the factors

- nonzeros only of the one column, in (row,value) pairs
- diagonal element first

Update of the factors

- E_{k+1} is an identity matrix except for $\boldsymbol{y}_p$ in column q

Modern Simplex Method

Given

$\boldsymbol{x}_{\mathcal{B}}$ (the basic solution)

a factorization of B suitable for computation

Choose an entering variable

solve $B^{\boldsymbol{T}}\boldsymbol{\pi} = \boldsymbol{c}_{\mathcal{B}}$

$p \in \mathcal{N}$: $c_p - \boldsymbol{\pi} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

solve $B\boldsymbol{y}_p = \boldsymbol{a}_p$

$q \in \mathcal{B}$: $x_q/y_{qp} = \min_{y_{ip}>0} x_i/y_{ip}$

Update

- update factorization to reflect change of basis
- update basic solution to $\boldsymbol{x}_{\mathcal{B}} - (x_q/y_{qp})\, \boldsymbol{y_p}$

1963

Linear Programming and Extensions

- George B. Dantzig

"... the *simplex algorithm* ... starts with a canonical form, consists of a sequence of pivot operations, and forms the main *subroutine* of the simplex method."

"Because some readers might find that the matrix notation of §8.5 [The Simplex Algorithm in Matrix Form] obscures the computational aspects, we have tended to avoid its use here."

LINEAR PROGRAMMING AND EXTENSIONS

by GEORGE B. DANTZIG

THE RAND CORPORATION

and

UNIVERSITY OF CALIFORNIA, BERKELEY

1963

PRINCETON UNIVERSITY PRESS

PRINCETON, NEW JERSEY

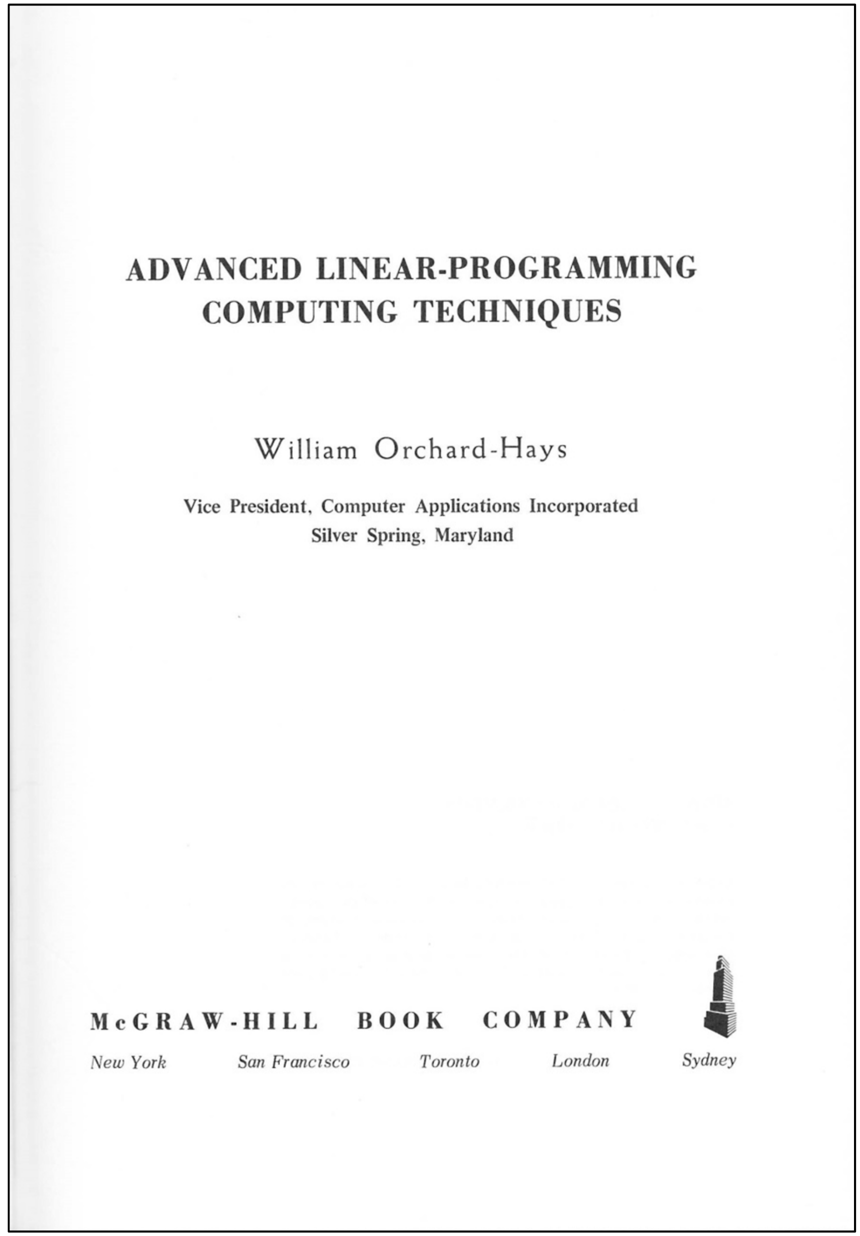
1968

Advanced Linear-Programming Computing Techniques

- William Orchard-Hays

"Except for [a few sections], the contents of the book reflect actual and extensive experience."

"I hope that the many users of mathematical programming systems implemented on today's large computers find the book valuable as background for the largely undocumented algorithms embedded in these systems. If it should also be found useful as a course text, all objectives will have been achieved."



ADVANCED LINEAR-PROGRAMMING
COMPUTING TECHNIQUES

William Orchard-Hays

Vice President, Computer Applications Incorporated
Silver Spring, Maryland

McGRAW-HILL BOOK COMPANY

New York San Francisco Toronto London Sydney

Tableau Simplex *Revisited*

Simple

- No linear algebra
- No matrices & inverses
- All computations in "pivot" step
- Easy to set up for hand calculation

Familiar

- Professors learned it
- Textbooks use it
- Proofs use it

But not inevitable . . .

Essential Simplex Method

Given

$\boldsymbol{x}_{\mathcal{B}}$ (the basic solution)

B (the basis)

Choose an entering variable

solve $B^{\boldsymbol{T}}\boldsymbol{\pi} = \boldsymbol{c}_{\mathcal{B}}$

$p \in \mathcal{N}$: $c_p - \boldsymbol{\pi} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

solve $B\boldsymbol{y}_p = \boldsymbol{a}_p$

$q \in \mathcal{B}$: $x_q / y_{qp} = \min\limits_{y_{ip} > 0} x_i / y_{ip}$

Update

- update basic solution to $\boldsymbol{x}_{\mathcal{B}} - (x_q / y_{qp})\, \boldsymbol{y_p}$