Automatic Reformulation of Second-Order Cone Programming Problems

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AMPL Optimization

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Second-order cone programming (SOCP)

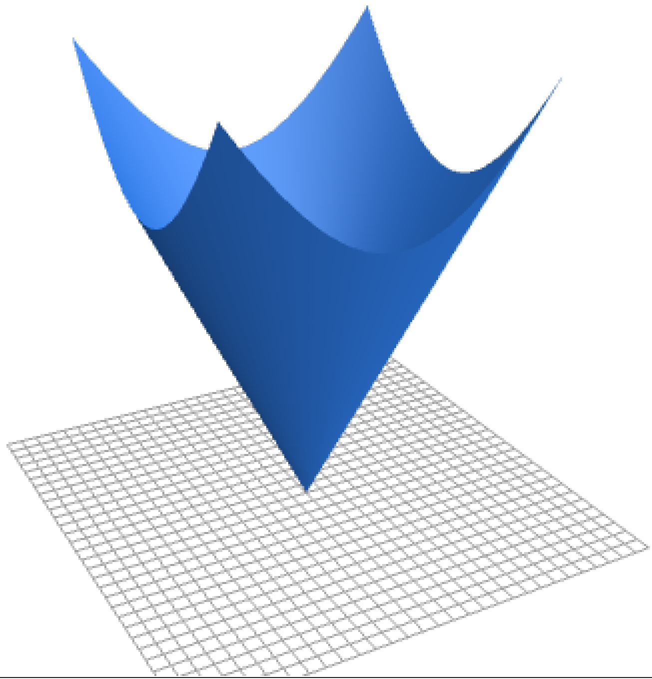
Problem statement:

$$
\begin{aligned}
\text{minimize} \quad & f^T x \\
\text{s.t.} \quad & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\
& Fx = g,
\end{aligned}
$$

where $x \in \mathbb{R}^n$ is the optimization variable,

$f \in \mathbb{R}^n, A_i \in \mathbb{R}^{n_i \times n}, b_i \in \mathbb{R}^{n_i}, c_i \in \mathbb{R}^n, d_i \in \mathbb{R},$
$F \in \mathbb{R}^{p \times n}$, and $g \in \mathbb{R}^p$.

SOCP constraint



$$\|Ax + b\|_2 \leq c^T x + d$$

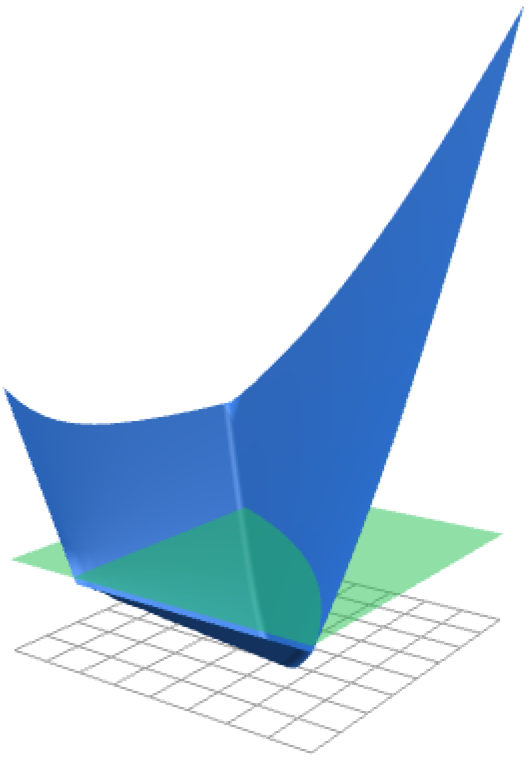
Motivation

Second-order cone programming problems

- Have a wide range of applications:
  - Robust optimization
  - Engineering applications: filter design, antenna array design, etc. See, for example, *Applications of Second-Order Cone Programming* by Lobo et al (1998).
- Can be solved efficiently with interior-point methods
- Many types of problems are convertible to SOCP (Erickson (2013))

But solvers only accept very limited forms of SOCP constraints

Example



$$\text{minimize}\sqrt{(x+2)^2+(y+1)^2}+\sqrt{(x+y)^2}$$

Problem

AMPL model:

```
var x;
var y;
minimize obj: sqrt((x + 2) ^ 2 + (y + 1) ^ 2) + sqrt((x + y) ^ 2);
```

CPLEX:

```
ampl: option solver cplex;
ampl: solve;
CPLEX 12.6.1.0: /tmp/at12668.nl contains a nonlinear objective.
```

MINOS:

```
ampl: option solver minos;
ampl: solve;
MINOS 5.51: Error evaluating objective obj: can't evaluate sqrt'(0).
ampl: let {i in 1.._nvars} _var[i] := 0.1; solve;
MINOS 5.51: optimal solution found?  Optimality
tests satisfied, but reduced gradient is large.
12 iterations, objective 2.19544929
Nonlin evals: obj = 126, grad = 125.
```

SOCP reformulation

$$
\begin{aligned}
\text{minimize} \quad & u + v \\
\text{s.t.} \quad & (x+2)^2 + (y+1)^2 \leq u^2, \\
& (x+y)^2 \leq v^2 \\
& u, v \geq 0
\end{aligned}
$$

Solving SOCP reformulation

AMPL model:

```
var x;
var y;
var u >= 0;
var v >= 0;
minimize obj: u + v;
s.t. c1: (x + 2) ^ 2 + (y + 1) ^ 2 <= u ^ 2;
s.t. c2: (x + y) ^ 2 <= v ^ 2;
```

CPLEX:

```
ampl: option solver cplex; solve;
CPLEX 12.6.1.0: QP Hessian is not positive semi-definite.
```

KNITRO:

```
ampl: option solver knitro; solve;
KNITRO 9.0.1: Locally optimal solution.
objective 2.121313302; feasibility error 9.97e-11
21 iterations; 22 function evaluations
```

Solver forms

"Standard" second-order cone constraint:

$$\sum_{i=1}^{n} a_i x_i^2 \leq a_{n+1} x_{n+1}^2$$

where $a_i \geq 0, x_{n+1} \geq 0$. Rotated cone constraint:

$$\sum_{i=1}^{n} a_i x_i^2 \leq a_{n+1} x_{n+1} x_{n+2}$$

where $a_i \geq 0, x_{n+1} \geq 0, x_{n+2} \geq 0$.

Making solver happy

AMPL model:

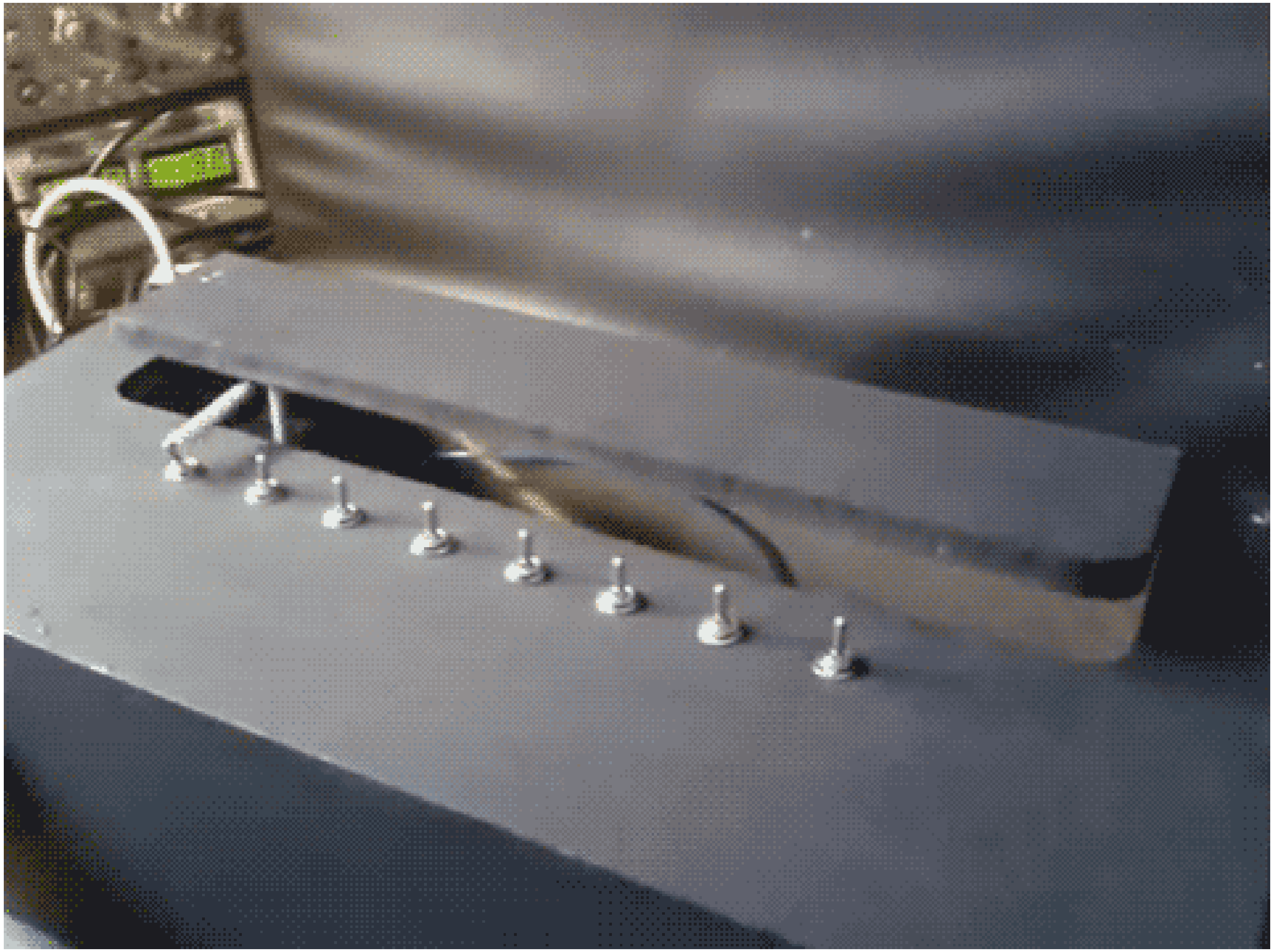
```
var x;
var y;
var u >= 0;
var v >= 0;
var r;
var s;
var t;
minimize obj: u + v;
s.t. c1: r ^ 2 + s ^ 2 <= u ^ 2;
s.t. c2: t ^ 2 <= v ^ 2;
s.t. c3: x + 2 = r;
s.t. c4: y + 1 = s;
s.t. c5: x + y = t;
```

CPLEX:

```
ampl: option solver cplex; solve;
CPLEX 12.6.1.0: optimal solution; objective 2.121320344
5 barrier iterations
```

Works but tedious and error-prone, so...

Let machine do the work

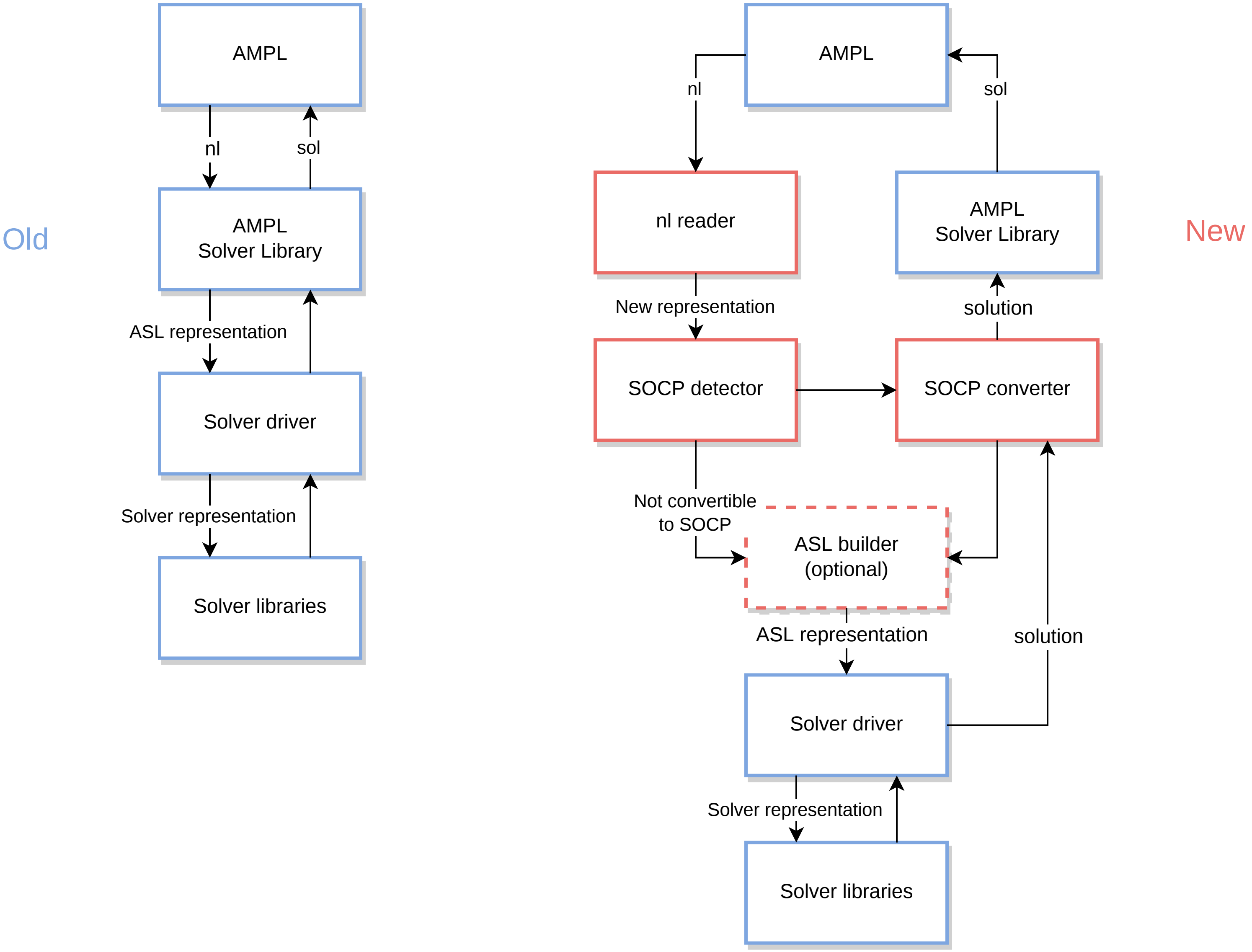


SOCP reformulation system

Features:

- Fast detection of problems convertible to SOCP
- Compatibility with existing solvers: no modifications to the source code of existing solvers required
- Automatic reformulation into SOCP forms accepted by solvers
- Easy to write new transformations
- Modular: components can be reused for different purposes

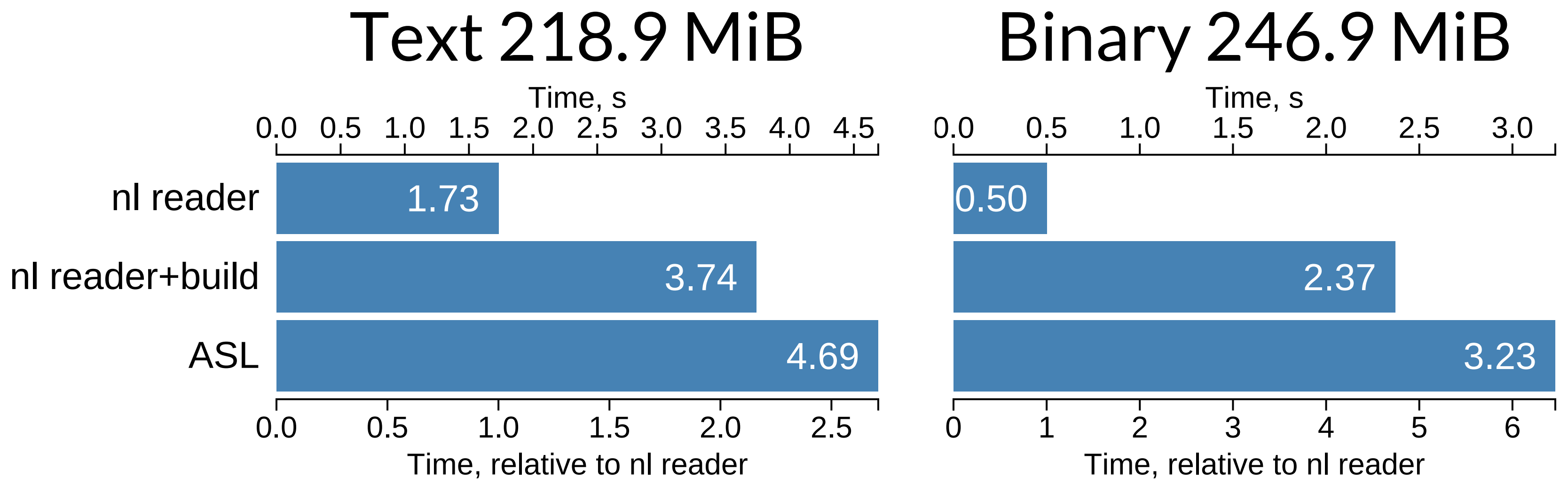
Architecture



nl reader

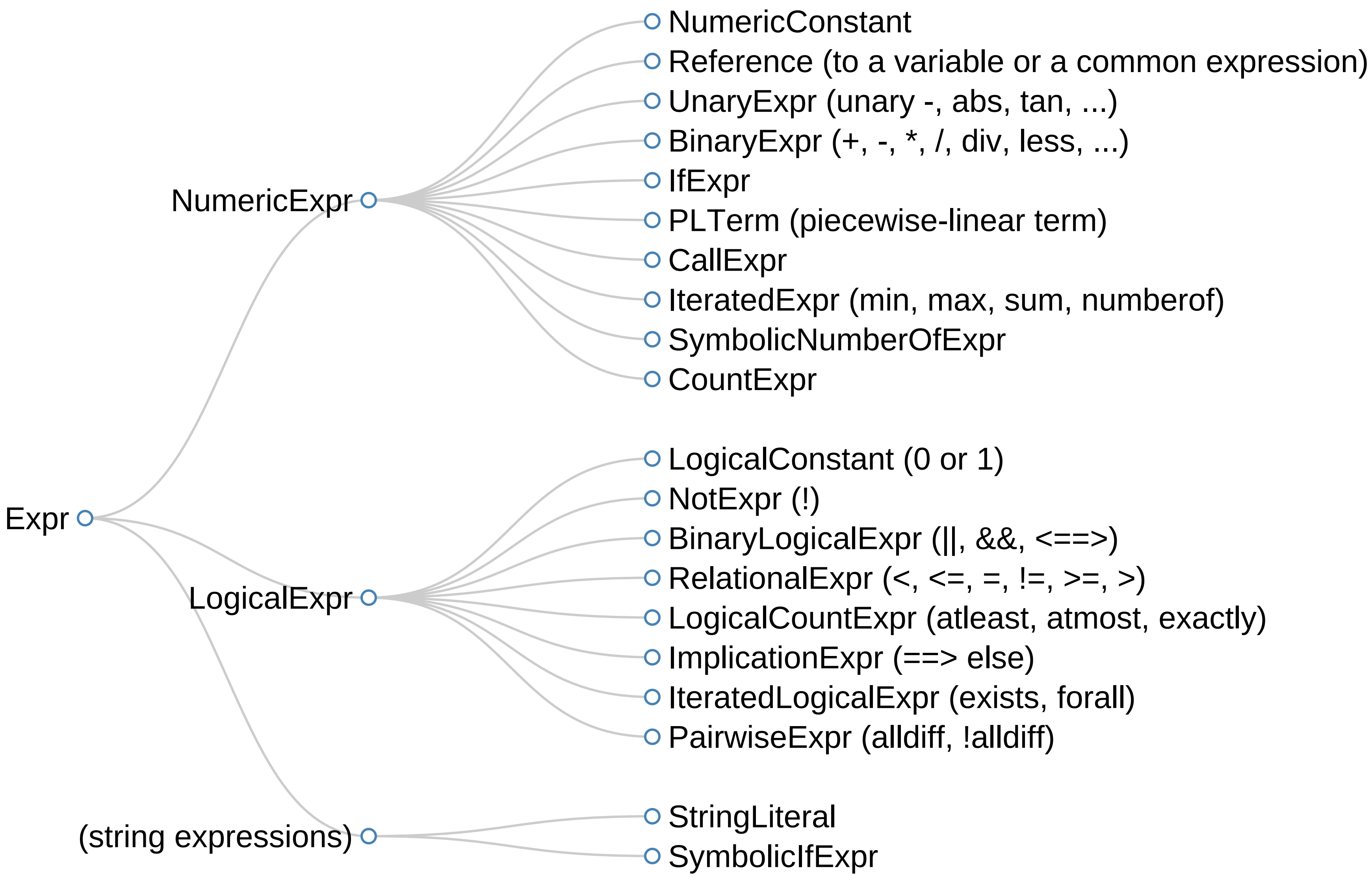
- High performance:
  - mmap-based
  - no dynamic memory allocations
  - handler methods can be inlined
- Simple SAX-like API
- Reusable: not limited to a single problem representation
- Complete

nl reader performance

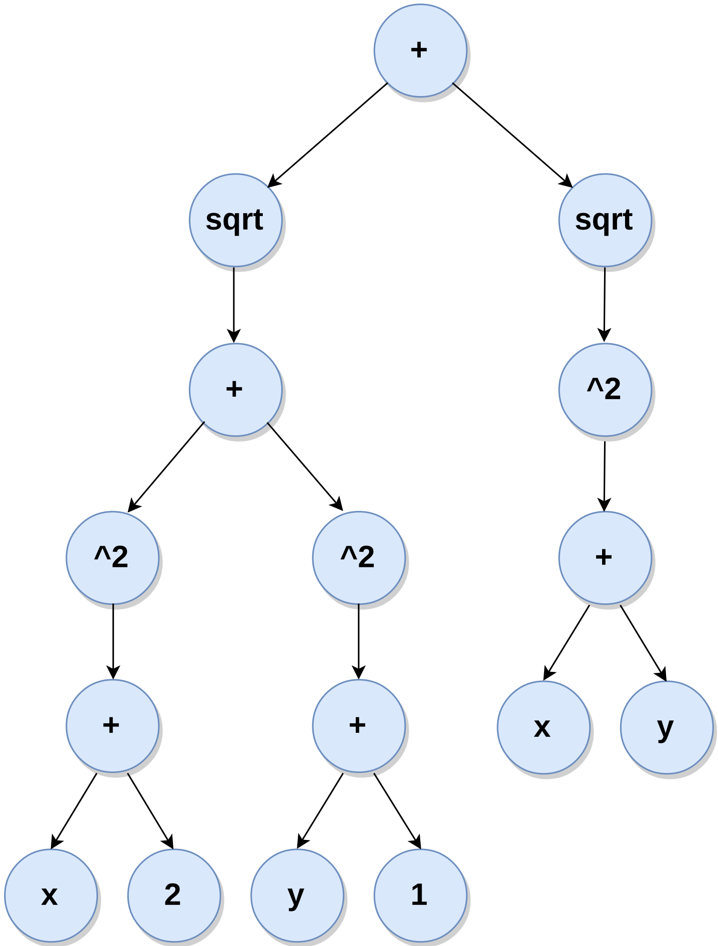


- 730 problems from the CUTE test set
- nl reader w/o problem construction is up to 6x faster than ASL
- Problem construction is faster than ASL, but has room for improvement (pool allocator)

Expression Classes

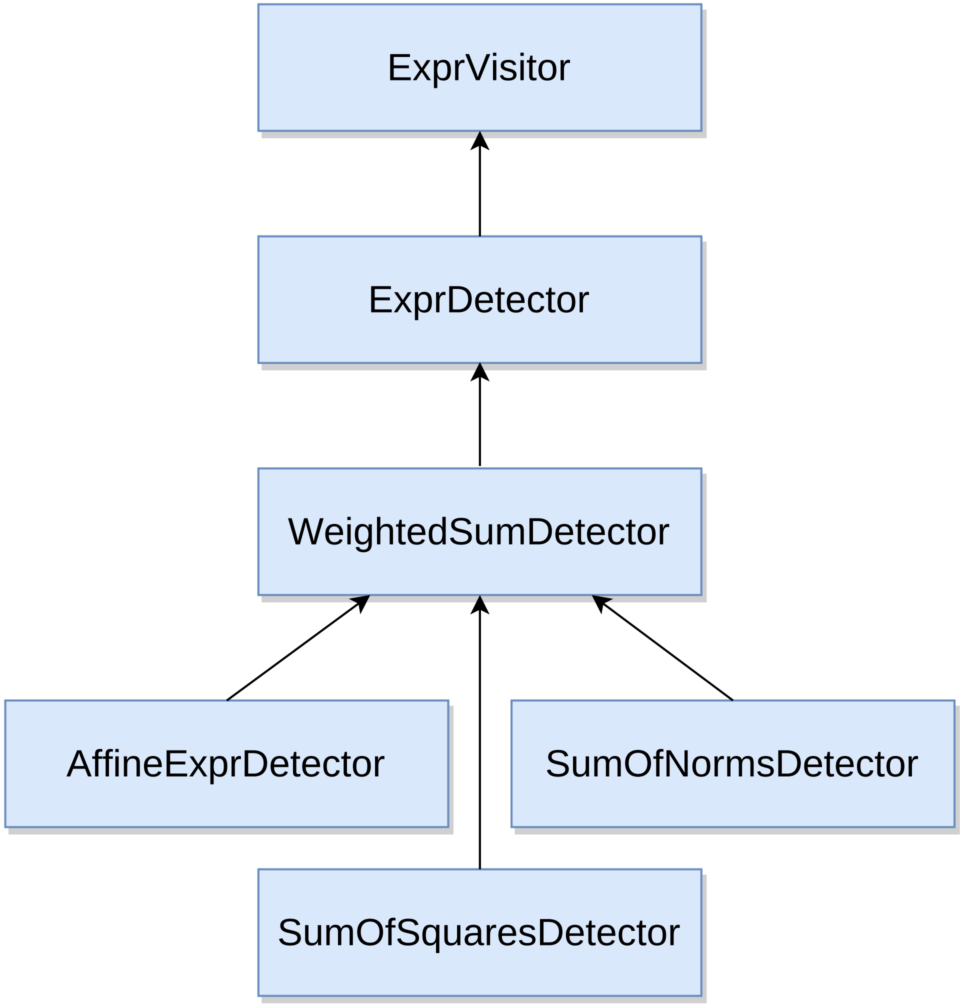


Expression Tree



$$\text{minimize} \sqrt{(x+2)^2 + (y+1)^2} + \sqrt{(x+y)^2}$$

Detectors



- Implemented as visitors
- Can be extended via inheritance
- Easy to implement
- Fast: no virtual calls, visit methods can be inlined
- Reusable

Weighted sum detector

Detects arbitrary combinations of sums and multiplications by positive constants:

```
template <typename Impl>
class WeightedSumDetector : public ExprDetector<Impl> {
 public:
  bool VisitAdd(BinaryExpr e) {
    return this->Visit(e.lhs()) && this->Visit(e.rhs());
  }
  bool VisitMul(BinaryExpr e) {
    return ((IsPosConstant(e.lhs()) && this->Visit(e.rhs())) ||
            (this->Visit(e.lhs()) && IsPosConstant(e.rhs())));
  }
  bool VisitSum(IteratedExpr e) {
    for (auto arg: e) {
      if (!this->Visit(arg)) return false;
    }
    return true;
  }
};
```

Affine expression detector

```
class AffineExprDetector :
  public WeightedSumDetector<AffineExprDetector> {
 public:
  bool VisitNumericConstant(NumericConstant) { return true; }
  bool VisitVariable(Variable) { return true; }
};
```

Sum of norms/squares detectors

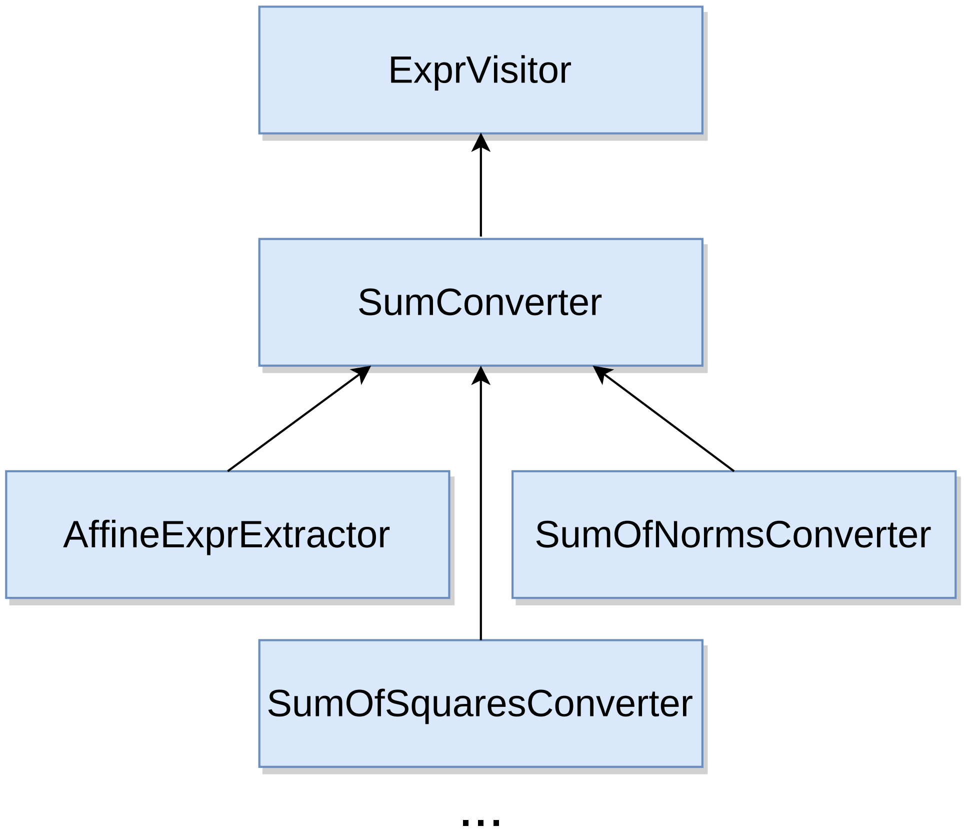
Detects sums of squares of affine expressions:

```
class SumOfSquaresDetector :
  public WeightedSumDetector<SumOfSquaresDetector> {
 public:
  bool VisitPow2(UnaryExpr e) {
    return AffineExprDetector().Visit(e.arg());
  }
};
```

Detects sums of norms:

```
class SumOfNormsDetector :
  public WeightedSumDetector<SumOfNormsDetector> {
 public:
  bool VisitSqrt(UnaryExpr e) {
    if (!SumOfSquaresDetector().Visit(e.arg()))
      return false;
    return true;
  }
};
```

Converters



- Implemented as visitors
- Mirror detectors class hierarchy
- More complicated than detectors, but still easy to implement
- Extensible, fast and reusable

Converters

- `SumConverter<Impl>`: recursively traverses sum (and multiplication by constant) expressions and applies conversion specified by `Impl` to the terms.
- `AffineExprExtractor`: extracts an affine expression e into a separate constraint $x = e$ where x is a new variable.
- `SumOfSquaresConverter`: replaces each term ce^2 with cx^2 where e is an affine expression and x is a variable in $x = e$ created by `AffineExprExtractor`.
- `SumOfNormsConverter`: replaces each term $\sqrt{e}$, where e is a sum of squares, with a new nonnegative variable x and adds a constraint $x^2 \geq e$.

Sum converter

```
template <typename Impl>
class SumConverter : public ExprVisitor<Impl, void> {
 private:
  Problem &problem_;
  double coef_;
 public:
  explicit SumConverter(Problem &p) : problem_(p), coef_(1) {}

  void VisitAdd(BinaryExpr e) {
    this->Visit(e.lhs());
    this->Visit(e.rhs());
  }
  void VisitSum(IteratedExpr e) {
    for (auto arg: e)
      this->Visit(arg);
  }
  ...
```

Sum of squares converter

```
class SumOfSquaresConverter :
  public SumConverter<SumOfSquaresConverter> {
 private:
  Problem::IteratedExprBuilder &sum_;

 public:
  SumOfSquaresConverter(Problem &p, Problem::IteratedExprBuilder &sum)
  : SumConverter<SumOfSquaresConverter>(p), sum_(sum) {}

  void VisitPow2(UnaryExpr e);
};
```

```
void SumOfSquaresConverter::VisitPow2(UnaryExpr e) {
  Problem &p = problem();
  AffineExprExtractor extractor(p);
  extractor.Apply(e.arg());
  // Replace the term ``coef * expr ^ 2`` with ``coef * x ^ 2``.
  NumericExpr term = p.MakeUnary(expr::POW2,
                                 p.MakeVariable(extractor.var_index()));
  if (coef() != 1)
    term = p.MakeBinary(expr::MUL, p.MakeNumericConstant(coef()), term);
  sum_.AddArg(term);
}
```

SOCP-convertible forms

Quadriatic constraints:

$$\sum_{i=1}^{n} a_i(\mathbf{f_i}\mathbf{x} + g_i)^2 \leq a_{n+1}(\mathbf{f_{n+1}}\mathbf{x} + g_{n+1})^2$$

where $a_i \geq 0$ and $\mathbf{f_{n+1}}\mathbf{x} + g_{n+1} \geq 0$ for all feasible $\mathbf{x}$.

$$\sum_{i=1}^{n} a_i(\mathbf{f_i}\mathbf{x} + g_i)^2 \leq a_{n+1}(\mathbf{f_{n+1}}\mathbf{x} + g_{n+1})(\mathbf{f_{n+2}}\mathbf{x} + g_{n+2})$$

where $a_i \geq 0, \mathbf{f_{n+1}}\mathbf{x} + g_{n+1} \geq 0, \mathbf{f_{n+2}}\mathbf{x} + g_{n+2} \geq 0$ for all feasible $\mathbf{x}$.

SOC-representable functions

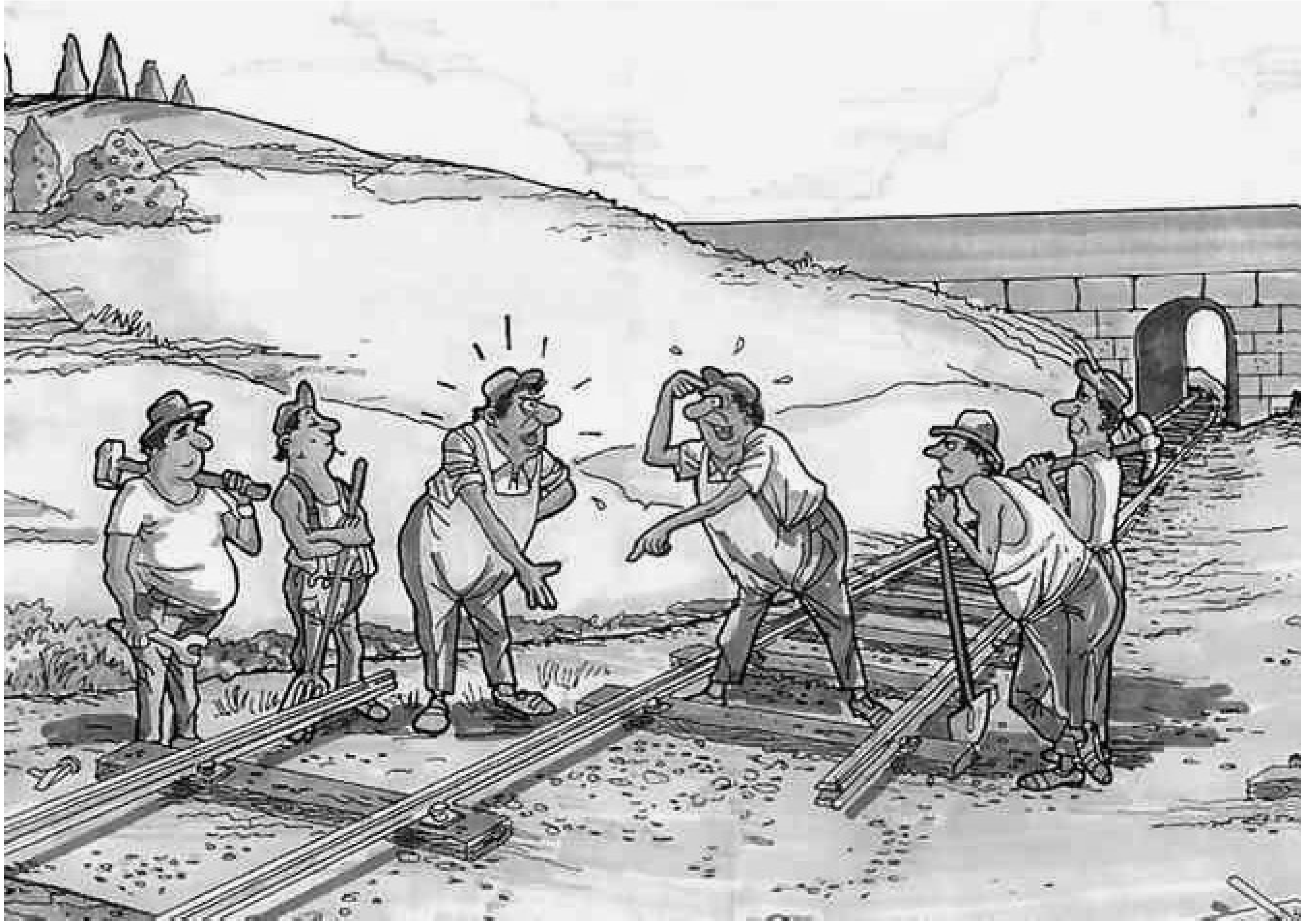
- A function $\mathrm{SOC}(x)$ is SOC-representable if $\mathrm{SOC}(\mathbf{x}) \leq \mathbf{f_{n+1}} \mathbf{x} + g_{n+1}$ can be equivalently represented by a collection of second-order cone and linear constraints.
- Any positive multiple, sum, or maximum of SOC-representable functions is also SOC-representable.
- Minimization of a SOC-representable function is equivalent to SOCP.

Examples of SOC-representable functions

- $(\sum_{i=1}^{n} a_i |\mathbf{f_i x} + g_i|^{\alpha_i})^{1/\alpha_0}$, where $\alpha_i \geq \alpha_0 \geq 1$. Includes norms.
- Quadratic-linear ratios: $\frac{\sum_{i=1}^{n} a_i (\mathbf{f_i x} + g_i)^2}{\mathbf{f_{n+2} x} + g_{n+2}}$
- Generalization of negative geometric mean: $\prod_{i=1}^{p} (\mathbf{f_i x} + g_i)^{-\alpha_i}$ for rational $\alpha_i \geq 0$.

and more (see Erickson (2013))

Integration



can be a challenge, proper design and planning is important.

Solver integration

- The following ASL functions are replaced using macros
  - `ASL_alloc` - allocates ASL data structure
  - `ASL_free` - frees ASL data structure
  - `jac0dim` - reads an nl file header
  - `qp_read` - reads the rest of an nl file
  - `write_sol` - writes a solution
- No changes to the driver source code, only build config - easy integration
- Can work with any AMPL solver that supports SOCP

Example revisited

AMPL model:

```
var x;
var y;
minimize obj: sqrt((x + 2) ^ 2 + (y + 1) ^ 2) + sqrt((x + y) ^ 2);
```

CPLEX*:

```
ampl: option solver cplex-socp;
ampl: solve;
CPLEX 12.4.0.0: optimal solution; objective 2.121320344
5 barrier iterations
No basis.
```

It just works!™

Summary

- **Current status**:
  - Reformulation infrastructure is ready
  - Transformations are being developed
- **Future work**:
  - More transformations
  - Build solver representation directly: faster, but requires modifications to a solver driver
  - Support more solvers: easy as only recompilation required

References

- Jared Erickson (2013). *Detection and Transformation of Objective and Constraint Structures for Optimization Algorithms*.
- Stephen Boyd, Lieven Vandenberghe (2004). *Convex Optimization*.
- Miguel Soma Lobo, Lieven Vandenberghe, Stephen Boyd, Hervé Lebret (1998). *Applications of Second-Order Cone Programming*.

- Source code: https://github.com/ampl/mp