How Linear Programming Became Practical

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IFORS 2021: The 22nd Conference of the International Federation of Operational Research Societies

Seoul, Korea *(online)* — 23-27 August 2021

The Evolution of Computationally Practical Linear Programming

Although a recognizable simplex approach to linear programming was being studied by Dantzig and others by 1947, the initially proposed algorithms were (and have remained) computationally impractical. Drawing on a series of obscure RAND technical reports, this talk tells the story of how the "revised" simplex method subsequently emerged to make today's powerful solvers possible. The presentation concludes by considering how the earlier, impractical simplex algorithms have come to be adopted by almost all textbooks, while computationally practical versions remain known mainly to experts.

1948

Programming of Interdependent Activities II: Mathematical Model

- George B. Dantzig
- *Econometrica* **17** (1949)

"Linear Programming"

I called my first paper: Programming in a Linear Structure. In the summer of 1948, Koopmans and I visited the RAND Corporation. One day we took a walk near the Santa Monica beach. Koopmans said: "Why not shorten Programming in a Linear Structure to Linear Programming?" I replied: "That's it! From now on that will be its name." Later that same day I save a talk at RAND entitled Linear Programming.

PROGRAMMING OF INTERDEPENDENT ACTIVITIES
II MATHEMATICAL MODEL[1]

BY GEORGE B. DANTZIG

Activities (or production processes) are considered as building blocks out of which a technology is constructed. Postulates are developed by which activities may be combined. The main part of the paper is concerned with the discrete type model and the use of a linear maximization function for finding the "optimum" program. The mathematical problem associated with this approach is developed first in general notation and then in terms of a dynamic system of equations expressed in matrix notation. Typical problems from the fields of inter-industry relations, transportation, nutrition, warehouse storage, and air transport are given in the last section.

INTRODUCTION

THE MULTITUDE of activities in which a large organization or a nation engages can be viewed not only as fixed objects but as representative building blocks of different kinds that might be recombined in varying amounts to form new blocks. If a structure can be reared of these blocks that is mutually self-supporting, the resulting edifice can be thought of as a technology. Usually the very elementary blocks have a wide variety of forms and quite irregular characteristics over time. Often they are combined with other blocks so that they will have "nicer" characteristics when used to build a complete system. Thus the science of programming, if it may be called a science, is concerned with the adjustment of the levels of a set of given activities (production processes) so that they remain mutually consistent and satisfy certain optimum properties.

It is highly desirable to have formal rules by which activities can be combined to form composite activities and an economy. These rules are set forth here as a set of postulates regarding reality. Naturally other postulates are possible; those selected have been chosen with a wide class of applications in mind and with regard to the limitations of present day computational techniques. The reader's attention is drawn to the last section of this report where a number of applications of the mathematical model are discussed. These are believed to be of sufficient interest in themselves, and may lend concreteness to the development which follows:

POSTULATES OF A LINEAR TECHNOLOGY

POSTULATE I: *There exists a set* $\{A\}$ *of activities.*
POSTULATE II: *All activities take place within a time span 0 to* t_0.

[1] A revision of a paper presented before the Madison Meeting of the Econometric Society on September 9, 1948. This is the second of two papers on this subject, both appearing in this issue. The first paper, with sub-title "General Discussion," will be referred to by Roman numeral I.

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1948

Programming of Interdependent Activities II: Mathematical Model

- George B. Dantzig
- *Econometrica* **17** (1949)

"Linear Programming"

- Formulations & applications
- No algorithm

"It is proposed to solve linear programming problems . . . by means of large scale digital computers . . . . Several computational procedures have been evolved so far and research is continuing actively in this field."

PROGRAMMING OF INTERDEPENDENT ACTIVITIES
II MATHEMATICAL MODEL[1]

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1949

Maximization of a Linear Function of Variables Subject to Linear Inequalities

- George B. Dantzig
- *Activity Analysis of Production and Allocation* (1951)

"Simplex Method"

The term Simplex Method arose out of a discussion with T. Motzkin who felt that the approach that I was using in the geometry of the columns was best described as a movement from one simplex to a neighboring one.

CHAPTER XXI

MAXIMIZATION OF A LINEAR FUNCTION OF VARIABLES SUBJECT TO LINEAR INEQUALITIES [1]

BY GEORGE B. DANTZIG

The general problem indicated in the title is easily transformed, by any one of several methods, to one which maximizes a linear form of nonnegative variables subject to a system of linear equalities. For example, consider the linear inequality $ax + by + c > 0$. The linear inequality can be replaced by a linear equality in nonnegative variables by writing, instead, $a(x_1 - x_2) + b(y_1 - y_2) + c - z = 0$, where $x_1 \geqq 0$, $x_2 \geqq 0$, $y_1 \geqq 0$, $y_2 \geqq 0$, $z \geqq 0$. The basic problem throughout this chapter will be considered in the following form:

PROBLEM: *Find the values of $\lambda_1, \lambda_2, \cdots, \lambda_n$ which maximize the linear form*

$$\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n \tag{1}$$

subject to the conditions that

$$\lambda_j \geqq 0 \qquad (j = 1, 2, \cdots, n) \tag{2}$$

and

$$\begin{aligned} \lambda_1 a_{11} + \lambda_2 a_{12} + \cdots + \lambda_n a_{1n} &= b_1, \\ \lambda_1 a_{21} + \lambda_2 a_{22} + \cdots + \lambda_n a_{2n} &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1 a_{m1} + \lambda_2 a_{m2} + \cdots + \lambda_n a_{mn} &= b_m, \end{aligned} \tag{3}$$

where a_{ij}, b_i, c_j are constants $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

[1] The author wishes to acknowledge that his work on this subject stemmed from discussions in the spring of 1947 with Marshall K. Wood, in connection with Air Force programming methods. The general nature of the "simplex" approach (as the method discussed here is known) was stimulated by discussions with Leonid Hurwicz.

The author is indebted to T. C. Koopmans, whose constructive observations regarding properties of the simplex led directly to a proof of the method in the early fall of 1947. Emil D. Schell assisted in the preparation of various versions of this chapter. Jack Laderman has written a set of detailed working instructions and has tested this and other proposed techniques on several examples.

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1949

Maximization of a Linear Function of Variables Subject to Linear Inequalities

- George B. Dantzig
- *Activity Analysis of Production and Allocation* (1951)

"Simplex Method"

- Proof of convergence
- No computers

> "As a practical computing matter the iterative procedure of shifting from one basis to the next is not as laborious as would first appear . . ."

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1953

An Introduction to Linear Programming

- W.W. Cooper, A. Henderson
- A. Charnes

"Simplex Tableau"

- Symbolic description
- Numerical example

"As far as computations are concerned it is most convenient to arrange the data at each stage in a 'simplex tableau' as shown in Table I.[12]"

"[12]A. Orden suggested this efficient arrangement developed by himself, Dantzig, and Hoffman."

An
INTRODUCTION
to
LINEAR
PROGRAMMING

PART I · An Economic Introduction
to Linear Programming
By W. W. COOPER and The Late A. HENDERSON

PART II · Lectures on the
Mathematical Theory
of Linear Programming
By A. CHARNES

NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

Terminology

Linear program

Minimize $\boldsymbol{c} \cdot \boldsymbol{x}$

Subject to $A\,\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

m constraints on n variables: $m < n$

Data

$\boldsymbol{b} = (b_1, \ldots, b_m)$

$\boldsymbol{c} = (c_1, \ldots, c_n)$

$A = [a_{ij}]$, with m rows $\boldsymbol{a^i}$ and n columns $\boldsymbol{a_j}$

Variables

$\boldsymbol{x} = (x_1, \ldots, x_n)$

Terminology *(cont'd)*

Basis

- $\mathcal{B}, \mathcal{N}$, sets of basic and nonbasic column indices
  - $|\mathcal{B}| = m, |\mathcal{N}| = n - m$
- $\boldsymbol{c}_{\mathcal{B}}, \boldsymbol{x}_{\mathcal{B}}$, corresponding subvectors of $\boldsymbol{c}, \boldsymbol{x}$

Basis matrix

- B, nonsingular $|\mathcal{B}| \times |\mathcal{B}|$ submatrix of A
- $B^{-1} = [z_{ij}]$, with $|\mathcal{B}|$ rows $\boldsymbol{z}^i$ and $|\mathcal{B}|$ columns $\boldsymbol{z}_j$

Tableau Simplex Method

Set up a $(|\mathcal{B}|+1)\times(|\mathcal{N}|+1)$ ***table of values***

$y_{ij},\ i\in\mathcal{B},\ j\in\mathcal{N}$: the transformed columns $\boldsymbol{y}_j = B^{-1}\boldsymbol{a}_j$

$y_{i0}\equiv x_i,\ i\in\mathcal{B}$: the basic solution $\boldsymbol{x}_\mathcal{B} = B^{-1}\boldsymbol{b}$

$y_{0j}\equiv d_j,\ j\in\mathcal{N}$: the reduced costs $c_j - \boldsymbol{c}_\mathcal{B}\boldsymbol{y}_j$

Choose an entering variable

$p\in\mathcal{N}$: $y_{0p} < 0$

Choose a leaving variable

$q\in\mathcal{B}$: $y_{q0}/y_{qp} = \min_{y_{ip}>0} y_{i0}/y_{ip}$

"Pivot" on the tableau

$y_{ij}\leftarrow y_{ij} - y_{qj}y_{ip}/y_{qp}$: subtracts multiples of row q from other rows

$y_{ip}\leftarrow -y_{ip}/y_{qp},\ y_{qp}\leftarrow 1/y_{qp}$

Impracticalities

Computational inefficiency

- $|\mathcal{B}| \times |\mathcal{N}| = m(n-m)$ additions & multiplications
- $|\mathcal{B}| \times |\mathcal{N}|$ numbers to write and store

Numerical instability

- Fixed rules for choosing p and q
- *risking* small denominators in $y_{ij} - y_{qj} y_{ip} / y_{qp}$
- *causing* loss of precision in pivot steps

1953

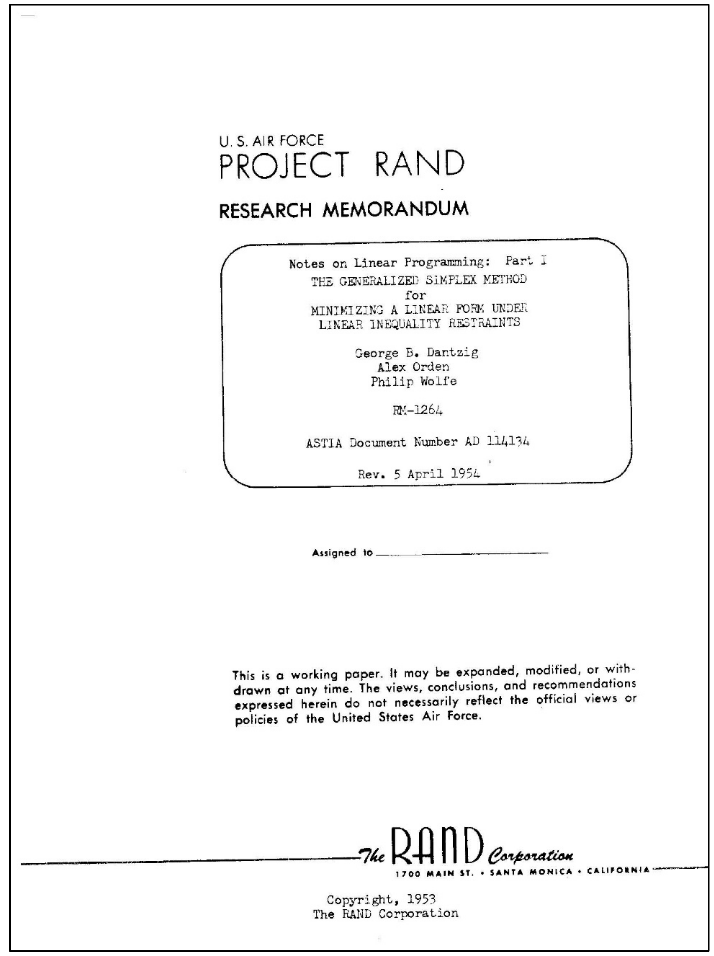
The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints

- George B. Dantzig, Alex Orden, Philip Wolfe
- *Project RAND Research Memorandum RM-1264*

"Lexicographic Simplex Method"

- Prevent cycling due to degeneracy
- Adapt computations accordingly

> "The k+1st iterate is closely related to the kth by simple transformations that constitute the computational algorithm [6], . . ."



U. S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

Notes on Linear Programming: Part I
THE GENERALIZED SIMPLEX METHOD
for
MINIMIZING A LINEAR FORM UNDER
LINEAR INEQUALITY RESTRAINTS

George B. Dantzig
Alex Orden
Philip Wolfe

RM-1264

ASTIA Document Number AD 114134

Rev. 5 April 1954

Assigned to ____________

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

1953

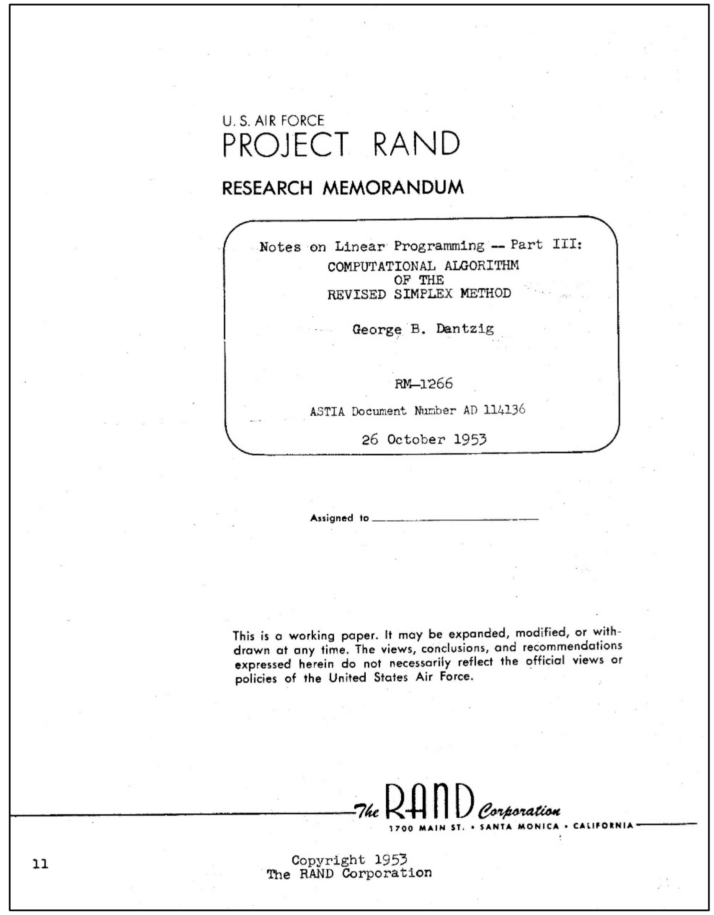
Computational Algorithm of the Revised Simplex Method

- George B. Dantzig
- *Project RAND Research Memorandum RM-1266*

"Revised Simplex Method"

- Tableau replaced by basis inverse
- Computations streamlined

"The transformation of just the inverse (rather than the entire matrix of coefficients with each cycle) has been developed because it has several important advantages over the old method: . . ."



U. S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

Notes on Linear Programming — Part III:
COMPUTATIONAL ALGORITHM
OF THE
REVISED SIMPLEX METHOD

George B. Dantzig

RM—1266

ASTIA Document Number AD 114136

26 October 1953

Assigned to ____________________

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11

Revised Simplex Method

Given a matrix of inverse values

$z_{ij},\ i \in \mathcal{B},\ j \in \mathcal{B}$: the basis inverse B^{-1}

$z_{i0} \equiv x_i,\ i \in \mathcal{B}$ (the basic solution)

$z_{0i} \equiv \pi_i,\ i \in \mathcal{B}$ (the dual prices)

Choose an entering variable

$p \in \mathcal{N}:\ c_p - \boldsymbol{z^0} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

$y_{ip} = \boldsymbol{z^i} \cdot \boldsymbol{a_p}$

$q \in \mathcal{B}:\ z_{q0}/y_{qp} = \min_{y_{ip}>0} z_{i0}/y_{ip}$

"Pivot" on the inverse

$z_{ij} \leftarrow z_{ij} - z_{qj}z_{ip}/z_{qp}$: subtracts multiples of row q from other rows

$z_{ip} \leftarrow -z_{ip}/z_{qp},\ z_{qp} \leftarrow 1/z_{qp}$

Advantages

Smaller update

"... In the original method (roughly) $m \times n$ new elements have to be recorded each time. In contrast, the revised method (by making extensive use of cumulative sums of products) requires the recording of about m^2 elements . . . ."

$$z_{ij} \leftarrow z_{ij} - z_{qj} z_{ip} / z_{qp}$$

Sparse operations

"In most practical problems the original matrix of coefficients is largely composed of zero elements. . . . The revised method works with the matrix in its original form and takes direct advantage of these zeros."

$$d_p = c_p - \boldsymbol{z^0} \cdot \boldsymbol{a_p}$$
$$y_{ip} = \boldsymbol{z^i} \cdot \boldsymbol{a_p}$$

Impracticalities

Inefficiency

- $|\mathcal{B}| \times |\mathcal{B}| = m^2$ additions & multiplications
- $|\mathcal{B}| \times |\mathcal{B}|$ numbers to write and store

Numerical instability

- Fixed rules for choosing p and q
- *risking* small denominators in $z_{ij} - z_{qj}z_{ip}/z_{qp}$
- *causing* loss of precision in pivot steps

However . . .

> "In contrast, the revised method (by making extensive use of cumulative sums of products) requires the recording of about m^2 elements (and an alternative method [5] can reduce this to m . . .)."

1953

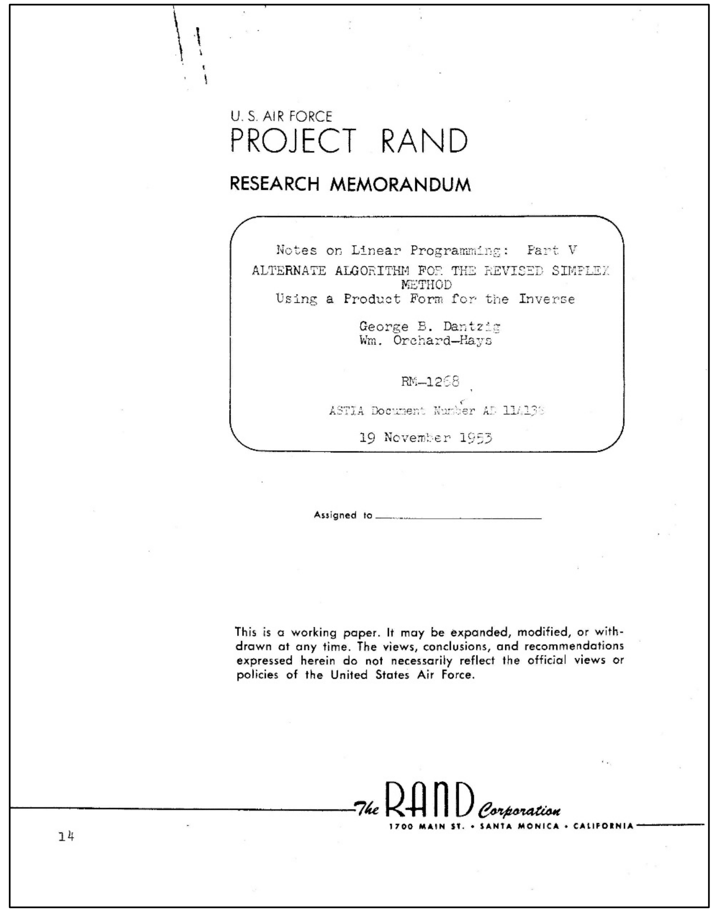
Alternate Algorithm for the Revised Simplex Method

- George B. Dantzig, Wm. Orchard-Hays
- *Project RAND Research Memorandum RM-1268*

"Product Form for the Inverse"

- Fully exploit sparsity of coefficients
- Solve practical problems

"Using the I.B.M. Card Programmed Calculator, . . . where the inverse matrix is needed at one stage and its transpose at another, this is achieved simply by turning over the deck of cards representing the inverse."



U. S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

Notes on Linear Programming: Part V
ALTERNATE ALGORITHM FOR THE REVISED SIMPLEX METHOD
Using a Product Form for the Inverse

George B. Dantzig
Wm. Orchard-Hays

RM-1268

ASTIA Document Number AD 114136

19 November 1953

Assigned to ______________________

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Product-Form Simplex Method

Given

$\boldsymbol{x}_{\mathcal{B}}$ (the basic solution)

$B^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$ (factorization of the basis inverse)

Choose an entering variable

$\boldsymbol{\pi} = \boldsymbol{c}_{\mathcal{B}} E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$

$p \in \mathcal{N}\colon\ c_p - \boldsymbol{\pi} \cdot \boldsymbol{a}_p < 0$

Choose a leaving variable

$\boldsymbol{y}_p = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1} \boldsymbol{a}_p$

$q \in \mathcal{B}\colon\ x_q / y_{qp} = \min_{y_{ip} > 0} x_i / y_{ip}$

Update

- add a factor E_{k+1}^{-1} derived from $\boldsymbol{y}_p$
- update the basic solution: $\boldsymbol{x}_{\mathcal{B}} \rightarrow \boldsymbol{x}_{\mathcal{B}} - (x_q / y_{qp})\, \boldsymbol{y}_p$

Factorization of the Inverse

Form of the factors

- E_i^{-1} is an identity matrix except for one column

1	0	0	1.7	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0.5	0	0
0	0	0	3.4	1	0
0	0	0	0	0	1

Computation of the factors

- Gauss-Jordan elimination
- Elimination ordering can be chosen to promote sparsity *and* stability

Storage of the factors

- *nonzeros only,* in (row,value) pairs
- diagonal element first

Update of the factors

- E_{k+1} is an identity matrix except for $\boldsymbol{y}_p$ in one column

Practical Simplex Method

Given

$\boldsymbol{x}_{\mathcal{B}}$ (the basic solution)

a factorization of B suitable for solving equations fast

Choose an entering variable

solve $B^{\boldsymbol{T}}\boldsymbol{\pi} = \boldsymbol{c}_{\mathcal{B}}$

$p \in \mathcal{N}\colon\ c_p - \boldsymbol{\pi} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

solve $B\boldsymbol{y}_p = \boldsymbol{a}_p$

$q \in \mathcal{B}\colon\ x_q / y_{qp} = \min_{y_{ip}>0} x_i / y_{ip}$

Update

- update factorization to reflect change of basis
- update basic solution to $\boldsymbol{x}_{\mathcal{B}} - (x_q / y_{qp})\ \boldsymbol{y_p}$

1963

Linear Programming and Extensions

- George B. Dantzig

"… the *simplex algorithm* … starts with a canonical form, consists of a sequence of pivot operations, and forms the main *subroutine* of the simplex method."

"Because some readers might find that the matrix notation of §8.5 [The Simplex Algorithm in Matrix Form] obscures the computational aspects, we have tended to avoid its use here."

LINEAR PROGRAMMING AND EXTENSIONS

by GEORGE B. DANTZIG

THE RAND CORPORATION

and

UNIVERSITY OF CALIFORNIA, BERKELEY

1963

PRINCETON UNIVERSITY PRESS

PRINCETON, NEW JERSEY

Tableau Simplex Method *Revisited*

Simple

- ❖ No linear algebra
- ❖ No matrices & inverses
- ❖ All computations in one “pivot” step
- ❖ Easy to set up for hand calculation

Familiar

- ❖ Textbooks presented it
- ❖ Students learned it
- ❖ Some students wrote new textbooks . . .

But still impractical

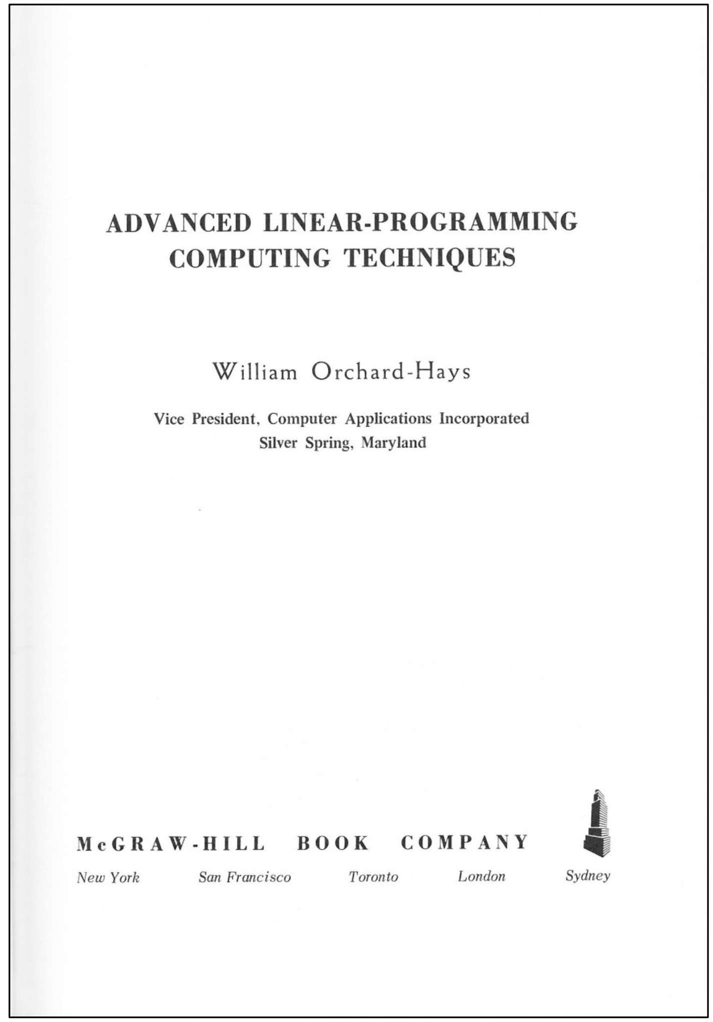
1968

Advanced Linear-Programming Computing Techniques

- William Orchard-Hays

"Except for [a few sections], the contents of the book reflect actual and extensive experience."

"I hope that the many users of mathematical programming systems implemented on today's large computers find the book valuable as background for the largely undocumented algorithms embedded in these systems. If it should also be found useful as a course text, all objectives will have been achieved."



ADVANCED LINEAR-PROGRAMMING COMPUTING TECHNIQUES

William Orchard-Hays

Vice President, Computer Applications Incorporated
Silver Spring, Maryland

McGRAW-HILL BOOK COMPANY
New York San Francisco Toronto London Sydney

Essential Simplex Method

Given

$\boldsymbol{x}_{\mathcal{B}}$ (the basic solution)

B (the basis)

Choose an entering variable

solve $B^{\boldsymbol{T}}\boldsymbol{\pi} = \boldsymbol{c}_{\mathcal{B}}$

$p \in \mathcal{N}$: $c_p - \boldsymbol{\pi} \cdot \boldsymbol{a_p} < 0$

Choose a leaving variable

solve $B\boldsymbol{y}_p = \boldsymbol{a}_p$

$q \in \mathcal{B}$: $x_q / y_{qp} = \min_{y_{ip} > 0} x_i / y_{ip}$

Update

- update basic solution to $\boldsymbol{x}_{\mathcal{B}} - (x_q / y_{qp})\, \boldsymbol{y_p}$

Essential Simplex Method

Course notes for use in teaching

- Optimization Methods I:
  Solving Linear Programs by the Simplex Method
- https://www.4er.org/CourseNotes

Slides Available at ampl.com

Recent and upcoming events

- https://ampl.com/resources/calendar/

News & events archive

- https://ampl.com/resources/learn-more/news-and-events-archive/

1978

History of Mathematical Programming Systems

- William Orchard-Hays
- *Design and Implementation of Optimization Software,* H.J. Greenberg, ed.

"Overview of an Era"

- Better implementations
- More powerful computers

"One cannot clearly comprehend the development of mathematical programming software without reference to the development of the computing field itself."

HISTORY OF MATHEMATICAL PROGRAMMING SYSTEMS

Wm. Orchard-Hays

International Institute for Applied Systems Analysis, Laxenburg, Austria

OVERVIEW OF AN ERA

One cannot clearly comprehend the development of mathematical programming software without reference to the development of the computing field itself. There are two main reasons, one specific and one general. First, mathematical programming and computing have been contemporary in an almost uniquely exact sense. Their histories parallel each other year by year in a remarkable way. Furthermore, mathematical programming simply could not have developed without computers. Although the converse is obviously not true, still linear programming was one of the important and demanding applications for computers from the outset. I will not try to trace early encouragement for the development of computers which emanated from influential agencies of the U.S. government and other quarters concerned with the application of LP and similar techniques. I have heard this story from unimpeachable sources but it antedates my personal experience and I might claim too much credit for our field. I am aware of later influences on computer technology for which we perhaps have not received sufficient credit. I will point out two or three of these along the way.

The second and more general reason for relating the two histories closely is based on the lessons of history itself. It is easy to find fault with the way things have developed in the past, whether political, cultural or technological. I predict that some of you will be tempted to ask during this two weeks, "But why did you do it that way, who not this way?" While it may be possible and even interesting to answer such questions--and I will try to anticipate some--it is largely futile to dwell on what may be

1978

History of Mathematical Programming Systems

- William Orchard-Hays
- *Design and Implementation of Optimization Software*

First, mathematical programming and computing have been contemporary in an almost uniquely exact sense. Their histories parallel each other year by year in a remarkable way.

Furthermore, mathematical programming simply could not have developed without computers. Although the converse is obviously not true, still linear programming was one of the important and demanding applications for computers from the outset.

The quarter century from the late 1940s to the early 1970s constituted an era, one of the most dynamic in the history of mankind. Among the many technological developments of that period — and indeed of any period — the computing field has been the most virulent and astounding.

. . . the nature of the computing industry, profession, and technology has by now been determined — all their essential features have existed for perhaps five years. One hopes that some of the more recent developments will be applied more widely and effectively but the technology that now exists is pretty much what will exist, leaving aside a few finishing touches to areas already well developed, such as minicomputers and networks.

1981

Reminiscences About the Origins of Linear Programming

- George B. Dantzig
- *Operations Research Letters* **1** (1982)

"Linear programming is viewed as a revolutionary development"

- System of linear inequalities
- Objective function
- Practical computational method

"Before closing let me tell some stories about how various linear programming terms arose."

Volume 1, Number 2 OPERATIONS RESEARCH LETTERS April 1982

REMINISCENCES ABOUT THE ORIGINS OF LINEAR PROGRAMMING *

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Received September 1981
Revised October 1981

The author recalls the early days of linear programming, the contributions of von Neumann, Leontief, Koopmans and others.

Linear Programming is viewed as a revolutionary development giving us the ability for the first time to state general objectives and to find, by means of the simplex method, optimal policy decisions for a broad class of practical decision problems of great complexity.

Linear programming, simplex method, history

Since its conception in 1947 in connection with the planning activities of the military, linear programming has come into wide use. In academic circles mathematicians, economists, and those who go by the name of Operations Researchers of Management Scientists, have written hundreds of books on the subject and, of course, an *unaccountable* number of articles.

Interestingly enough, in spite of its wide applicability to everyday problems, linear programming was unknown prior to 1947. It is true that two or three individuals may have been aware of its potential—for example Fourier in 1823 and de la Vallee Poussin in 1911. But these were isolated cases. Their works were soon forgotten. Kantorovich in 1939 made an extensive proposal that was neglected by the U.S.S.R. It was only after the major developments in mathematical programming had taken place in the West that Kantorovich's paper became known around 1959. To give some idea of how meager the research effort was: Motzkin in his Ph.D. thesis lists only 42 papers before 1936 on linear inequality systems by such authors as Stokes, Dines, McCoy and Farkas.

My own contributions to the field grew out of my World War II experience. I had become an expert on programming planning methods using desk calculators. In 1946 I was the Mathematical Advisor to the U.S. Air Force Comptroller. I had just formally completed my Ph.D. and was looking for an academic position. In order to entice me into *not* taking another job, colleagues challenged me to see what could be done to mechanize the planning process. I was asked to find a way to compute more rapidly a time-staged deployment, training and logistical supply *program*. In those days mechanization meant using analog devices or punch card equipment.

Consistent with my training as a mathematician, I set out to formulate a model. I was fascinated by the work of Wassily Leontief who proposed in 1932 a simple matrix structure which he called the *Interindustry Input–Output Model* of the American Economy. It was simple in concept and could be implemented in sufficient detail to be useful for practical planning. I soon saw that it had to be generalized. Leontief's was a steady-state model and what was needed was a highly *dynamic model*, one that could change over time. In Leontief's model there was a one-to-one correspondence between the production processes and the items produced by these processes. What was needed was a model with many *alternative activities*. The application was to be *large scale* with hundreds of items and activities. Finally it had to be *computable*. Once the model was formulated, there had to be a practical way to compute what quantities of these activities to engage in that was consistent with their respective input–output

* This research was partially supported by the Department of Energy Contract AM03-76SF00326, PA No. DE-AT03-76ER72018; Office of Naval Research Contract N00014-75-C-0267; National Science Foundation Grants MCS-7681259, MCS-7926009 and ECS-8012974.

0167-6377/82/0000–0000/$02.75 © 1982 North-Holland